VII GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN
THE CORRESPONDENCE ROUND. SOLUTIONS

1. (A.Zaslavsky) (8) Does a convex heptagon exist which can be divided into 2011 equal triangles?

   Solution. Yes, it does. For example, let T be a right-angled triangle such that its hypothenuse is equal to 1003 and one of its legs is equal to 1. Consider rectangles formed by two such triangles and compose a rectangle with side equal to 1003 from such rectangles. To one of these sides, attach an isosceles triangle composed from two triangles equal to T, and to the second side attach a quadrilateral composed from three such triangles (fig.1).

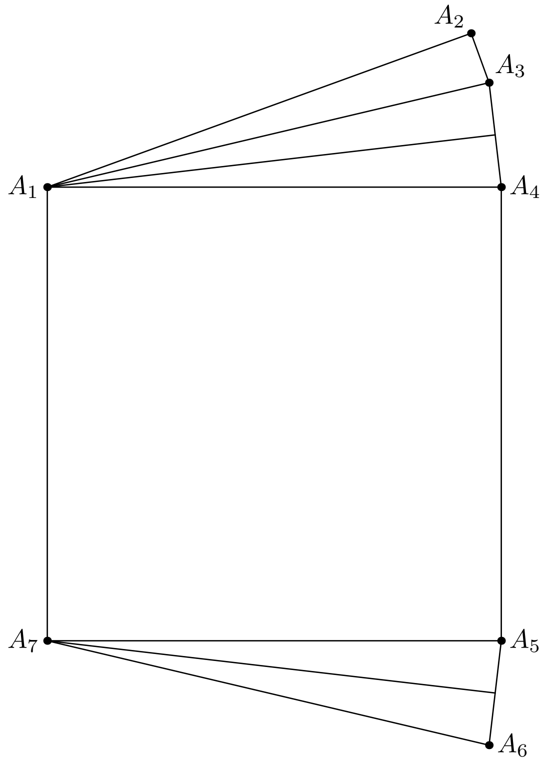

Fig. 1

   Second solution. Take a square with side equal to 34 and from three its angles cut isosceles right-angled triangles with legs equal to 3, 6 and 16. The obtained heptagon can be divided into isosceles right-angled triangles with legs equal to 1, and the number of these triangles is equal to $2 \cdot 34^2 - 9 - 36 - 256 = 2011$.

   Also we can take a rectangle $m \times n$ and cut from it isosceles right-angled triangles with legs x, y, z such that $2mn - x^2 - y^2 - z^2 = 2011$. The participants of olympiad found several such solutions.

   Third solution. (M.Amanzholov, Kazahstan) Let T be an isosceles triangle with the base equal to 1 and with the angle at the opposite vertex equal to 120°. Then the regular triangle with side equal to 1 can be divided into three such triangles. On the other hand, from 335 regular triangles we can compose an isosceles trapezoid with bases equal to 168 and 167 and lateral sides equal to 1. Two such trapezoids with common greatest base form a convex hexagon which can be divided into 2010 triangles equal to T. Joining such triangle to its smallest side, we construct the sought heptagon.

2. (From Singapore olympiads) (8) Let ABC be a triangle with sides $AB = 4$, $AC = 6$. Point H

is the projection of vertex B to the bisector of angle A. Find MH, where M is the midpoint of BC.

Solution. Let D be the common point of BH and AC (fig.2). Then AH is the bisector and the altitude of triangle ABD. Thus this triangle is isosceles, i.e. $AD = BD$ and $BH = HD$. So MH ic the medial line of triangle BCD and $MH = CD/2 = (AC - AB)/2 = 1$.

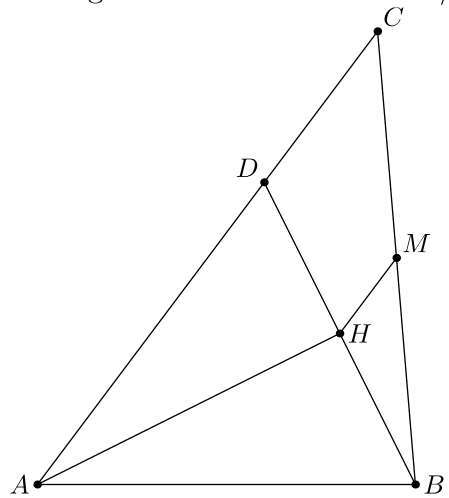

Fig. 2

3. (D.Shvetsov) (8) Let ABC be a triangle with $\angle A = 60°$. The midperpendicular of segment AB meets line AC at point C_1. The midperpendicular of segment AC meets line AB at point B_1. Prove that line B_1C_1 touches the incircle of triangle ABC.

   Solution. Let B_0, C_0 be the midpoints of AC, AB respectively. Since triangles AB_0B_1, AC_0C_1 are right-angled with $\angle A = 60°$, we have $AB_1 = 2AB_0 = AC$ and $AC_1 = 2AC_0 = AB$. Thus line B_1C_1 is the reflection of BC in the bisector of angle A. Since this bisector passes through the incenter and BC touches the incircle, B_1C_1 also touches the incircle.

4. (B.Frenkin) (8) Segments AA', BB', CC' are the bisectrices of triangle ABC. It is known that these lines are also the bisectrices of triangle $A'B'C'$. Is it true that triangle ABC is regular?

   Solution. Yes, the diagonal CC' of quadrilateral $A'C'B'C$ is the bisector of its angles C and C', thus it is the symmetry axis. Then $A'C = B'C$, $A'C' = B'C'$, $\angle CB'A' = \angle CA'B'$ and $\angle AB'C' = \angle BA'C'$. Similarly $\angle BC'A' = \angle BA'C' = \angle AB'C' = \angle AC'B'$. Thus triangles $AB'C'$ and $BA'C'$ are equal, i.e. $AB' = BA'$ and $AC = BC$. Similarly we obtain that $AB = BC$.

5. (B.Frenkin) (8) Given triangle ABC. The midperpendicular of side AB meets one of the remaining sides at point C'. Points A' and B' are defined similarly. For which original triangles triangle $A'B'C'$ is regular?

   Answer. For regular triangles and triangles with angles equal to 30, 30 and 120 grades.

   Solution. Consider a non-regular triangle ABC. Let AB be its greatest side. Then points A', B' lie on segment AB. From the condition of the problem we obtain that $C'C_0$, where C_0 is the midpoint of AB, is the bisector of segment $A'B'$, thus $CA' = A'B = AB' = CB'$, i.e. C' coincides with C and triangle ABC is isosceles. Also we have $2\angle A = \angle A + \angle CAB' = \angle CB'B = 60°$, so $\angle A = \angle B = 30°$.

6. (A.Akopjan) (8) Two unit circles ω_1 and ω_2 intersect at points A and B. M is an arbitrary point of ω_1, N is an arbitrary point of ω_2. Two unit circles ω_3 and ω_4 pass through both points M and N. Let C be the second common point of ω_1 and ω_3, and D be the second common point of ω_2 and ω_4. Prove that $ACBD$ is a parallelogram.

   Solution. Let O_i be the center of circle ω_i. From the condition of the problem we obtain that O_1AO_2B, O_1CO_3M, O_3MO_4N, O_4NO_2D are rhombuses with sides equal to 1. Then $\overrightarrow{O_1C} = \overrightarrow{MO_3} = \overrightarrow{O_4N} = \overrightarrow{DO_2}$ and $\overrightarrow{O_1A} = \overrightarrow{BO_2}$. Thus $\overrightarrow{AC} = \overrightarrow{DB}$, q.e.d.

7. (A.Akopjan) (8–9) Points P and Q on sides AB and AC of triangle ABC are such that $PB = QC$. Prove that $PQ < BC$.

   Solution. Let T be the fourth vertex of parallelogram $CBPT$. Then $PT = BC$ and $CT = BP = CQ$ (fig.7). Hence $\angle PQT > \angle TQC = \angle QTC > \angle QTP$, i.e. $PT > PQ$.

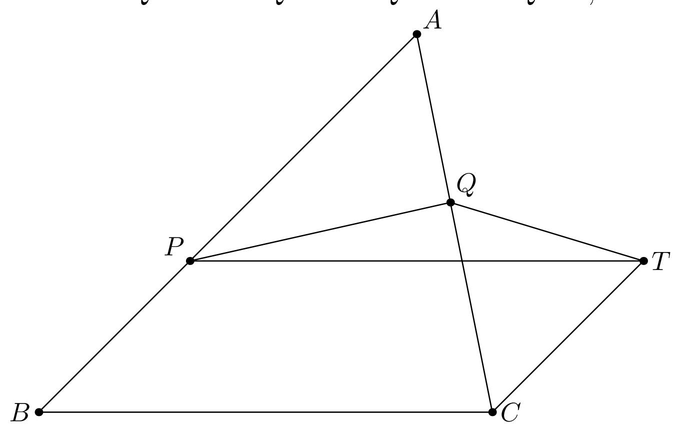

Рис. 7

8. (D.Shvetsov) (8–9) The incircle of right-angled triangle ABC ($\angle B = 90°$) touches AB, BC, CA at points C_1, A_1, B_1 respectively. Points A_2, C_2 are the reflections of B_1 in lines BC, AB respectively. Prove that lines A_1A_2 and C_1C_2 meet on the median of triangle ABC.

   Solution. Let I be the incenter and P be the common point of line A_1A_2 with median BB_0 (fig.8). Since $\angle IA_1P = \angle IA_1B_1 = \angle C/2 = \angle PBA_1/2$, then $\angle BA_1P = \angle BPA_1$, i.e. $BP = BA_1$. Since $BA_1 = BC_1$, line C_1C_2 also passes through P.

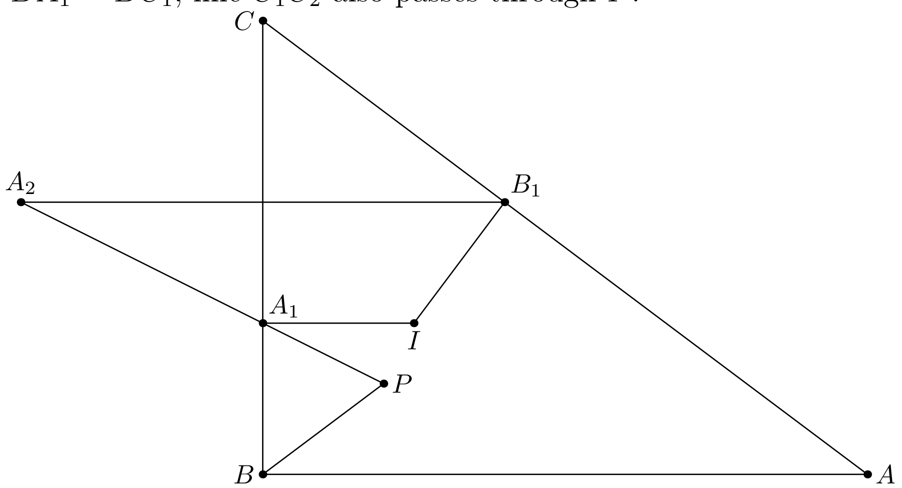

Fig. 8

9. (D.Shvetsov) (8–9) Let H be the orthocenter of triangle ABC. The tangents to the circumcircles

of triangles CHB and AHB at point H meet AC at points A_1 and C_1 respectively. Prove that $A_1H = C_1H$.

Solution. We have from the condition of the problem that $\angle AHC_1 = \angle ABH$. Thus $\angle C_1HB' = \angle AHB' - \angle ABH = \angle HAB = \pi/2 - \angle ABC$ (B' is the base of the altitude). Similarly $\angle B'HA_1 = \pi/2 - \angle ABC$, i.e triangle A_1HC_1 is isosceles.

10. (M.Volchkevich) (8–9) The diagonals of trapezoid $ABCD$ meet at point O. Point M of lateral side CD and points P, Q of bases BC and AD are such that segments MP and MQ are parallel to the diagonals of the trapezoid. Prove that line PQ passes through point O.

Solution. By Thales theorem $AQ/QD = AM/MB = CP/PB$. Thus $AQ/PC = AD/BC = AO/CO$. Then triangles AOQ and COP are similar and $\angle AOQ = \angle COP$.

11. (D.Shvetsov) (8–10) The excircle of right-angled triangle ABC ($\angle B = 90°$) touches side BC at point A_1 and touches line AC at point A_2. Line A_1A_2 meets the incircle of ABC for the first time at point A'; point C' is defined similarly. Prove that $AC||A'C'$.

Solution. Let I be the incenter and PQ be the diameter of the incircle parallel to AC (fig.11). Since $\angle PIC = \angle ACI = \angle BCI$ and $CA_1 = (AB + BC - AC)/2 = r = IP$, quadrilateral IPA_1C is an isosceles trapezoid. Then line A_1P is parallel to IC, i.e it coincides with A_1A_2. So P coincides with A', and similarly Q coincides with C'.

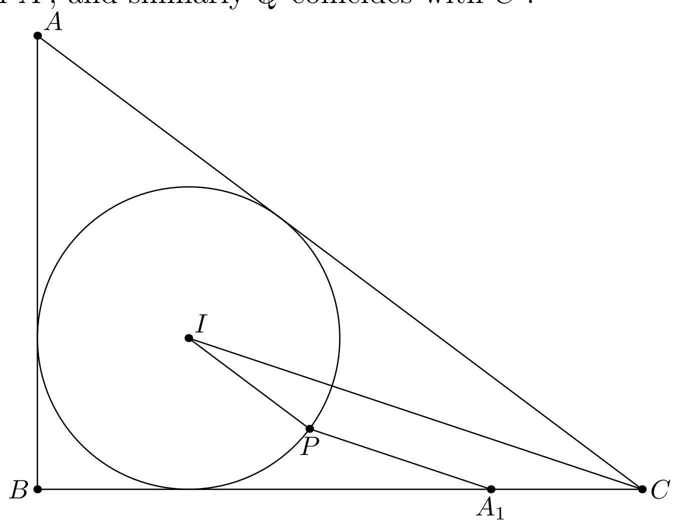

Fig.11

12. (V.Yasinsky) (8–10) Let AP and BQ be altitudes of acute-angled triangle ABC. Using a compass and a ruler, construct a point M on side AB such that $\angle AQM = \angle BPM$.

Solution. Since points P, Q lie on the circle with diameter AB, $\angle BPQ = 180° - \angle A$. Then $\angle MPQ = \angle BPQ - \angle BPM = 180° - \angle A - \angle AQM = \angle AMQ$. Thus the circle passing through points P, Q, M touches AB (fig.12).

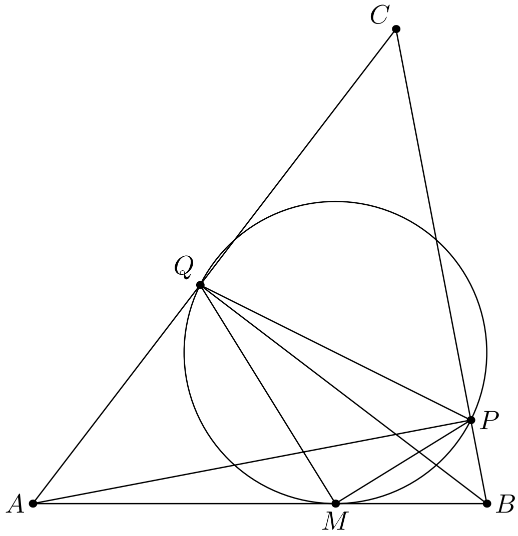

Fig.12

13. (B.Frenkin) a) (8–10) Find the locus of centroids for triangles whose vertices lie on the sides of a given triangle (each side contains a single vertex).

    b) (11) Find the locus of centroids for tetrahedrons whose vertices lie on the faces of a given tetrahedron (each face contains a single vertex).

    Solution. a) Let points A', B', C' lie on sides BC, CA, AB of triangle ABC. Since the midpoint C_0 of segment $A'B'$ lies inside triangle ABC, the distance from C_0 to AB is shorter than the respective altitude of ABC. Since the centroid M of $A'B'C'$ divides segment $C'C_0$ as $2:1$, the distance from M to AB is less than $2/3$ of this altitude. Similarly the distances from M to two remaining sidelines are less than $2/3$ of respective altitudes, i.e. M lies inside a hexagon formed by the sides of triangle and its reflections in the centroid of ABC. When two vertices of $A'B'C'$ tend to the same vertex of ABC, M tends to the boundary of this hexagon. Thus all its inner points belong to the sought locus.

    b) Similarly to a) we obtain that the sought locus is the polyhedron bounded by the faces of the given tetrahedron and by the planes parallel to the faces and dividing the respective altitudes in ratio $1:3$ starting from the vertex. Four of eight faces of this polyhedron are triangles, and four remaining faces are hexagons.

14. (B.Frenkin) (9) In triangle ABC, the altitude and the median from vertex A form (together with line BC) a triangle such that the bisectrix of angle A is the median; the altitude and the median from vertex B form (together with line AC) a triangle such that the bisectrix of angle B is the bisectrix. Find the ratio of sides for triangle ABC.

    Solution. Since the bisector of angle B bisects the angle between the altitude and the median, angle B is right. Thus the altitude from A coincides with AB, i.e. the bisector of angle A divides BC in ratio $1:3$. Then ratio $AB:AC$ also is equal to $1:3$ and by Pythagoras theorem $BC:AB = 2\sqrt{2}$.

15. (V.Protasov) (9–10) Given a circle with center O and radius equal to 1. AB and AC are the tangents to this circle from point A. Point M on the circle is such that the areas of quadrilaterals $OBMC$ and $ABMC$ are equal. Find MA.

    Solution. Since $S_{OBMC} - S_{ABMC} = S_{ABC} - S_{OBC} + 2S_{MBC}$, the locus of points for which $S_{OMBC} = S_{AMBC}$, is the bisector of segment OA. Thus $AM = OM = 1$.

16. (P.Dolgirev) (9–10) Given are triangle ABC and line l. The reflections of l in AB and AC meet at point A_1. Points B_1, C_1 are defined similarly. Prove that

a) lines AA_1, BB_1, CC_1 concur;

b) their common point lies on the circumcircle of ABC;

c) two points constructed in this way for two perpendicular lines are opposite.

Solution. Firstly note that when l moves remaining parallel to itself with constant velocity, the reflections of l in AC and BC also move with constant velocities. Then C_1 moves along the line passing through C, i.e the common point of CC_1 with the circumcircle depends only on the direction of l. Now let A', B' be the common points of l with BC and AC (fig.16). Then $\angle C_1B'C = \angle CB'A'$, $\angle C'AC = \angle BA'C_1$. Thus C is the incenter or the excenter of triangle $A'B'C_1$, i.e. C_1C bisects angle $A'C_1B'$ or the adjacent angle. But the angle between lines $A'C_1$ and $B'C_1$ doesn't depend on l, thus the angle between CC_1 and C_1A' also doesn't depend on l. So when l rotates, lines AA_1, BB_1, CC_1 rotate with the same velocity. This yields all assertions of the problem.

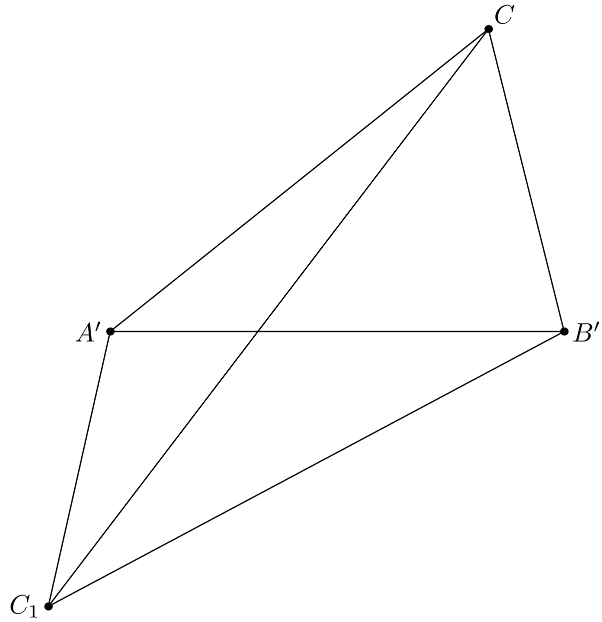

Fig.16

17. (B.Frenkin) (9–11) a) Does there exist a triangle in which the shortest median is longer that the longest bisectrix?

b) Does there exist a triangle in which the shortest bisectrix is longer that the longest altitude?

Solution. a) No, it doesn't. Let the lengths of sides BC, AC, AB be equal to a, b, c respectively and $a \le b \le c$. Now let CM be the median, AL be the bisector. If angle C isn't acute then $AL > AC$. Since $BC \le AC$, $\angle CMA$ isn' acute, thus $CM \le AC$ and $CM < AL$.

Now let $\angle C$ be acute. Since AB is the greatest side, $\angle C \ge 60^\circ$ and angles A, B are acute. Then the base H of altitude AH lies on segment BC. Thus AH (and AL) isn't less than $AC\cos\ 60^\circ = b\sqrt{3}/2$. But the square of CM is equal to $\frac{2a^2+2b^2-c^2}{4} \le \frac{2a^2+b^2}{4} \le \frac{3b^2}{4}$. So CM isn't greater than $b\sqrt{3}/2$ and can't exceed the bisector of angle A.

b) No, it doesn't. Let $a \le b \le c$ and l be the bisector of angle C. Then $(al + bl)\sin\frac{C}{2} = 2S_{ABC} = ab\sin C$, i.e. $l = \frac{2ab\cos\frac{C}{2}}{a+b}$. On the other hand, the altitude from A is equal to $h = b\sin C$. Since

$a + b \geq 2a$ and $C \geq 60^\circ$, $h/l = (a+b)\sin\frac{C}{2}/a \geq 1$.

Note. It is easy to construct a triangle such that its shortest median is longer than its longest altitude.

18. (A.Zaslavsky) (9–11) On the plane, given are n lines in general position, i.e. any two of them aren't parallel and any three of them don't concur. These lines divide the plane into several parts. What is

    a) the minimal;

    b) the maximal

    number of these parts that can be angles?

    Solution. a) **Answer.** 3. Consider the convex envelope of all common points of given lines. Two lines passing through some vertex of this envelope divide the plane into four angles, and one of them contains all the remaining points. Thus the remaining lines don't intersect the vertical angle and the number of angles can't be less than three. The example with three angles can be constructed by induction: the line in turn must intersect all previous lines inside the triangle which is the convex envelope of common points.

    b) **Answer.** n if n is odd, $n-1$ if $n > 2$ is even. Construct a circle containing all common points. The given lines divide it into $2n$ arcs. Let AB, BC be two adjacent arcs, X, Y be the common points of the line passing through B, with the lines passing through A and C respectively. If X lies on segment BY then the part containing arc BC can't be an angle. Thus only one of two parts containing adjacent arcs can be an angle. Thus the number of angles isn't greater than n, and an equality is possible only when the part containing each second arc is an angle. But if n is even, this yields that there exist two angles containing the opposite arcs. Since these two angles are formed by the same lines, this isn't possible if $n > 2$. If n is odd then n parts formed by the sidelines of a regular n-gon are angles. It is evident that we can add one line without reduction of the number of angles.

    Second solution of a). (A.Goncharuk, Kharkov) Let polygon T be the union of all bounded parts. Then all angles are vertical to the angles of T, which are less than 180°. From the formula for the sum of angles we obtain that there exist three such angles. The polygon with three angles can be constructed in the following way. Take point D inside triangle ABC, inscribe a circle with sufficiently small radius in angle ADB and take $n-4$ points on the smaller arc formed by touching points. The tangents in these points and lines AC, BC, AD, BD form the sought polygon.

19. (A.Zaslavsky) (9–11) Does there exist a nonisosceles triangle such that the altitude from one vertex, the bisectrix from the second one and the median from the third one are equal?

    Solution. Yes, it does. Fix vertices A, B, construct point D which is the reflection of A in B, and consider an arbitrary point C such that $\angle BCD = 150^\circ$. The altitude of triangle ABC from A is equal to the distance DH from D to BC, i.e. $CD/2$. The median BM from B as the medial line in triangle ACD also is equal to $CD/2$ (fig.19). Now move point C along arc BD, containing angle 150°. When C tends to B, the bisector from C tends to zero and the median from B tends to $AB/2$. When C tends to D, the median from B tends to zero and the bisector isn't less than BC. Thus there exists a point C for which the bisector is equal to two remaining segments.

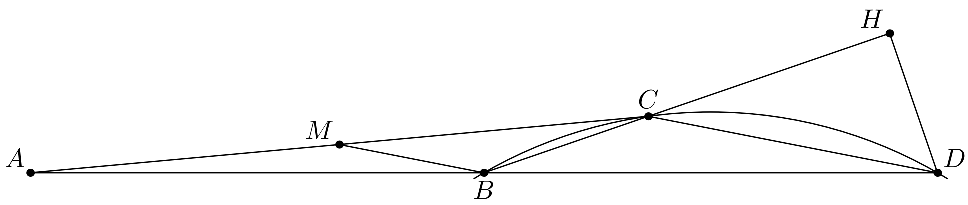

Fig.19

Note. When C moves from B to D, the bisector increases and the altitude decreases. Thus the angles of the sought triangle can be determined uniquely.

20. (N.Beluhov, A.Zaslavsky) (9–11) Quadrilateral $ABCD$ is circumscribed around a circle with center I. Points M and N are the midpoints of diagonals AC and BD. Prove that $ABCD$ is a cyclic quadrilateral if and only if $IM : AC = IN : BD$.

    Solution. Consider the case when $ABCD$ isn't a trapezoid. The other case needs no essential changes.

    By Newton theorem I lies on MN. Let $\lambda = MI : IN$. On the sides, construct points P, Q, R and S such that $AP : PB = CQ : QB = CR : RD = DS : SA = \lambda$. Let us prove that I is the midpoint of segments PR and QS.

    Use the masses method: place the unit masses in points A, C, and the masses equal to λ in points B, D. Replacing two first masses by the mass 2 in point M, and two remaining masses by the mass 2λ in point N, we obtain that I is the masscenter of four masses. On the other hand, we can replace the masses in A and B by the mass $1 + \lambda$ in point P and two remaining masses by the same mass in point R.

    Now since I is the midpoint of PR, and lines AB and CD are non-parallel tangents to the circle with center I, the angles formed by these lines with PR are equal i.e. PR is parallel to the bisector of some angle formed by these lines. Similarly QS is parallel to the bisector of some angle formed by AD and BC. Thus $ABCD$ is cyclic iff $PR \perp QS$. Since $PQRS$ is a parallelogram (with side parallel to AC and BD), this yields that $PQRS$ is a rhombus. But $PQ = QR \Leftrightarrow \frac{1}{1+\lambda}AC = \frac{\lambda}{1+\lambda}BD \Leftrightarrow \lambda = AC : BD$, q.e.d.

21. (V.Yasinsky) (10–11) On a circle with diameter AC, let B be an arbitrary point distinct from A and C. Points M, N are the midpoints of chords AB, BC, and points P, Q are the midpoints of smaller arcs related to these chords. Lines AQ and BC meet at point K, and lines CP and AB meet at point L. Prove that lines MQ, NP and KL concur.

    Solution. Lines PM and QN meet in the center O of the circle. Thus it suffices to apply the Desargues theorem to triangles PML and NQK.

22. (G.Feldman) (10–11) Let CX, CY be the tangents from vertex C of triangle ABC to the circle passing through the midpoints of its sides. Prove that lines XY, AB and the tangent at point C to the circumcircle of ABC concur.

    Solution. The homothety with center C and factor $1/2$ transforms line XY to the radical axis of point C and the circle passing through the midpoints A', B', C' of BC, CA, AB. On the other hand, the tangent at C to the circumcircle touches also the circle $A'B'C'$, i.e. it is the radical axis of this circle and point C. The common point of these radical axes lies on $A'B'$. Using the inverse homothety we obtain the assertion of the problem.

23. (N.Beluhov, M.Marinov) (10–11) Given are triangle ABC and line l intersecting BC, CA and AB at points A_1, B_1 and C_1 respectively. Point A' is the midpoint of the segment between the projections of A_1 to AB and AC. Points B' and C' are defined similarly.

(a) Prove that A', B' and C' lie on some line l'.

(b) Suppose l passes through the circumcenter of $\triangle ABC$. Prove that in this case l' passes through the center of its nine-points circle.

Solution. Let P_a, P_b, P_c be the midpoints of altitudes AH_a, BH_b, CH_c. It is evident that A', B', C' lie on the sidelines of triangle $P_aP_bP_c$ and divide its sides in the same ratios as points A_1, B_1, C_1 divide the sides of ABC. Thus a) immediately follows from Menelaos theorem. In addition, if l passes through some fixed point, then l' passes also through some fixed point, so it suffices to prove b) for two lines passing through the circumcenter O. Let us consider for example the lines passing through a vertex of ABC.

Let C_1 be the common point of CO and AB; X, Y be the projections of C_1 to AC and BC; A_0, B_0, C_0 be the midpoints of BC, CA, AB; U, V be the midpoints of XY and A_0B_0; Q be the common point of the bisector to segment A_0B_0 and of line UP_c (fig.23). Since $XY \parallel AB$, points C, V, U are collinear. Then $VQ/CP_c = UV/UC = C_1O/CC_1$, i.e. $VQ = OC_0/2$ and Q is the circumcenter of $A_0B_0C_0$.

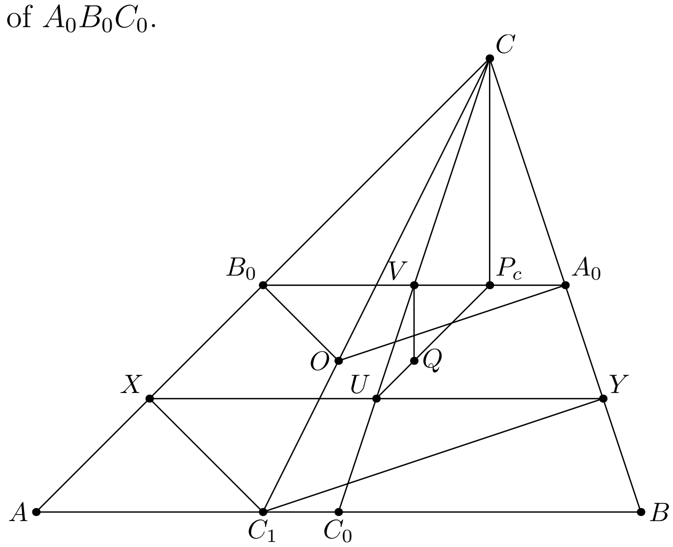

Fig.23

24. (A.Zaslavsky) (10–11) Given is an acute-angled triangle ABC. On sides BC, CA, AB, find points A', B', C' such that the longest side of triangle $A'B'C'$ is minimal.

Solution. Firstly let us prove that the sought triangle is pedal, i.e three perpendiculars from its vertices to respective sides of ABC concur. Indeed for an arbitrary triangle $A'B'C'$ circles $AB'C'$, $BC'A'$ and $CA'B'$ have some common point P. Let A'', B'', C'' be the projections of P to BC, CA, AB. Since $\angle A'PB' = \angle A''PB'' = \pi - \angle C$ etc., $\angle A''PA' = \angle B''PB' = \angle C''PC'$, and triangle $A''B''C''$ can be obtained from $A'B'C'$ by spiral similarity with center P and the factor less than 1.

Consider now point T with regular pedal triangle and prove that the pedal triangle of any point P has at least one greater side. Let A', B' be the projections of P to BC and AC. Then $A'B' = PC \sin C$, i.e. $A'B'$ isn't greater than the side of pedal triangle of T iff $PC \leq TC$. Similarly we obtain $PB \leq TB$, $PA \leq TA$. It is clear that these three inequalities can be correct only for point T.

Finally construct point T. From its definition we have $TA \cdot BC = TB \cdot AC = TC \cdot AB$. First equality defines the Apollonius circle passing through C and the bases of internal and external bisectors of angle C. Similarly the second equality defines the second circle. T is the common point of these circles lying inside triangle ABC.

25. (N.Beluhov) (10–11) Three equal regular tetrahedrons have the common center. Is it possible that all faces of the polyhedron that forms their intersection are equal?

    Solution. Yes, it is possible. Let the first tetrahedron touch their common inscribed sphere at points A, B, C, D. Rotate these points around the common perpendicular (and bisector) of the segments AB and CD by 120° to obtain A', B', C', D' and by 240° to obtain A'', B'', C'', D'' (the twelve points form two regular hexagons). The tangential planes to the sphere in these twelve points form the three tetrahedrons needed. Indeed, for any two of these points there exists an isometry that maps this set of twelve points onto itself and maps one of these two points to another one. These isometries enable us to map any facet of the obtained polygon onto any other one.