VIII GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN
THE CORRESPONDENCE ROUND. SOLUTIONS

1. (M.Rozhkova) (8) In triangle ABC point M is the midpoint of side AB, and point D is the foot of altitude CD. Prove that $\angle A = 2\angle B$ if and only if $AC = 2MD$.

   Solution. Let K be the midpoint of AC (fig.1). Since DK is the median of a right-angled triangle ADC, we obtain that $AK = KD$ and $\angle ADK = \angle A$. On the other hand, MK is a medial line of ABC, therefore, $\angle DMK = \angle B$. Applying the external angle theorem to triangle DMK we obtain that the equalities $KD = DM$ and $\angle KDA = 2\angle KMD$ are equivalent.

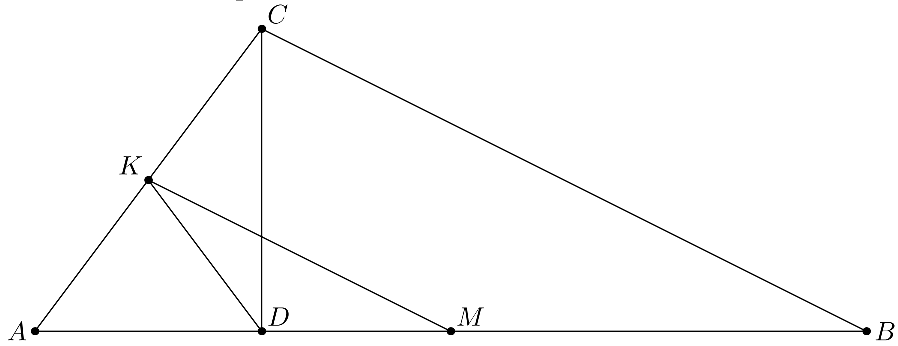

Fig.1

2. B.Frenkin) (8) A cyclic n-gon is divided by non-intersecting (inside the n-gon) diagonals to $n-2$ triangles. Each of these triangles is similar to at least one of the remaining ones.

   For what n this is possible?

   Answer. For $n = 4$ and for $n > 5$.

   Solution. It is clear that $n > 3$. Now if n is even then we can bisect a regular n-gon to two equal polygons by a diagonal passing through its center and divide these two polygons by the same way. Also we can construct on three sides of a regular $2k$-gon equal triangles with vertices on the circumcircle. Thus for odd $n > 5$ such situation is also possible. Prove that it isn't possible for $n = 5$.

   If the circumcenter of a pentagon doesn't lie on dividing diagonals then the triangle containing it is acute-angled and two remaining triangles are obtuse-angled, i.e. the condition of the problem can't be true. If the circumcenter lies on the diagonal then two triangles adjacent with this diagonal are right-angled and the third triangle is obtuse-angled. Thus the condition also isn't true.

3. (D.Shvetsov) (8) A circle with center I touches sides AB, BC, CA of triangle ABC in points C_1, A_1, B_1. Lines AI, CI, B_1I meet A_1C_1 in points X, Y, Z respectively. Prove that $\angle YB_1Z = \angle XB_1Z$

   Solution. Since $B_1I \perp AC$, it is sufficient to prove that $\angle YB_1A = \angle XB_1C$. Since CI is the medial bisector to A_1B_1, therefore $\angle YB_1A_1 = \angle C_1A_1B_1$, and since $\angle A_1B_1C = \angle B_1A_1C$, therefore $\angle YB_1A = \angle C_1A_1B$ (fig.3). Similarly $\angle XB_1C = \angle A_1C_1B = \angle C_1A_1B$.

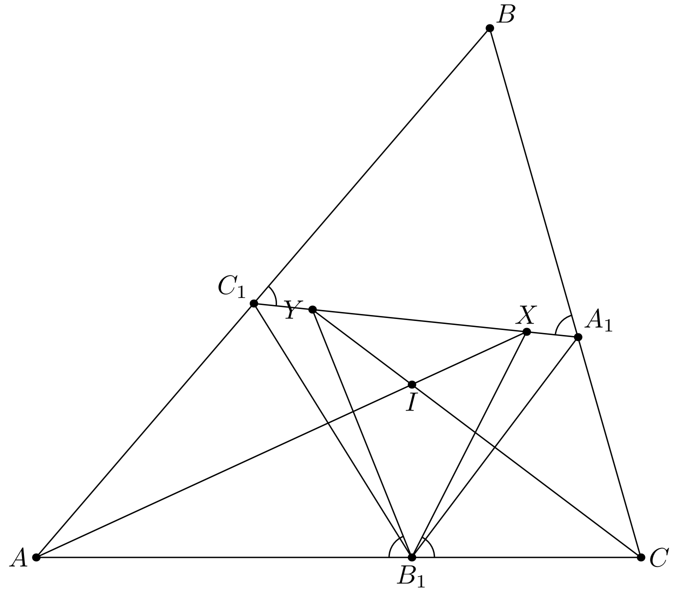

Fig.3

4. (A.Akopyan) (8) Given triangle ABC. Point M is the midpoint of side BC, and point P is the projection of B to the perpendicular bisector of segment AC. Line PM meets AB in point Q. Prove that triangle QPB is isosceles.

   Solution. Let D be the reflection of B in the medial bisector to AC, and T be the common point of AB and CD. Then $ACBD$ is an isosceles trapezoid, thus BDT is an isosceles triangle (fig.4). The line PM contains the medial line of this triangle, Therefore triangle QPB is also isosceles.

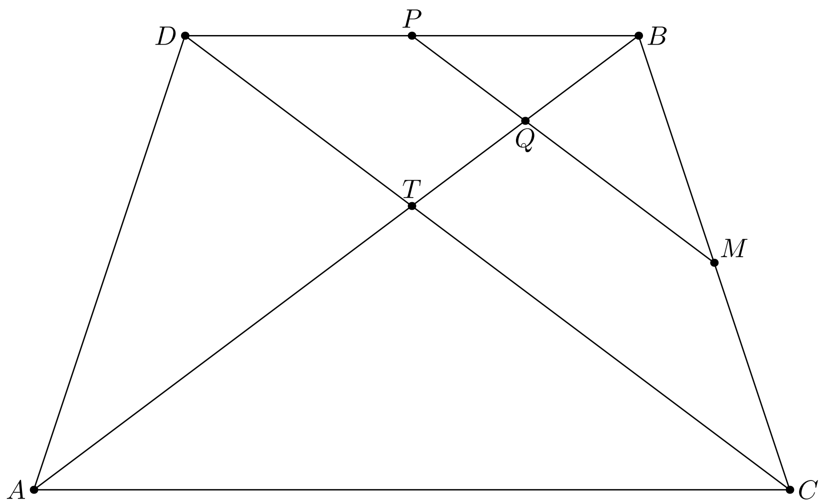

Fig.4

5. (D.Shvetsov) (8) On side AC of triangle ABC an arbitrary point is selected D. The tangent in D to the circumcircle of triangle BDC meets AB in point C_1; point A_1 is defined similarly. Prove that $A_1C_1||AC$.

   Solution. The condition yields that $\angle C_1DA = \angle DBC$ and $\angle A_1DC = \angle DBA$ (fig.5). Therefore A_1BC_1D is a cyclic quadrilateral, i.e. $\angle C_1A_1D = \angle C_1BD = \angle CDA_1$.

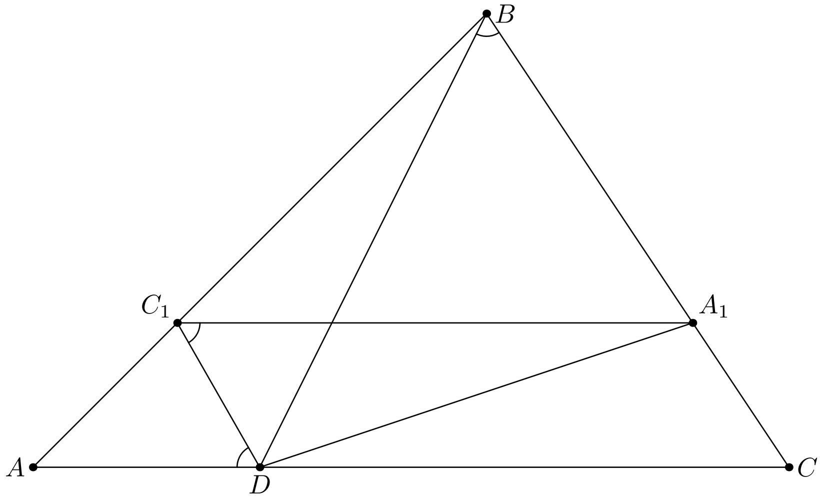

Fig.5

6. (D.Shvetsov) (8–9) Point C_1 of hypothenuse AC of a right-angled triangle ABC is such that $BC = CC_1$. Point C_2 on cathetus AB is such that $AC_2 = AC_1$; point A_2 is defined similarly. Find angle AMC, where M is the midpoint of A_2C_2.

   Answer. $135°$.

   Solution. Let I be the incenter of ABC. Since C_1 is the reflection of B in CI, and C_2 is the reflection of C_1 in AI, we obtain that $BI = IC_2$ and $\angle BIC_2 = 90°$. Similarly $BI = IA_2$ and $\angle BIA_2 = 90°$ (fig.6). Therefore, I is the midpoint of A_2C_2, and $\angle AIC = 135°$.

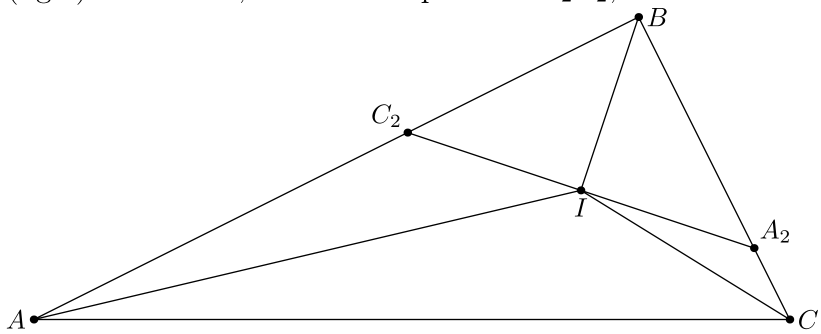

Fig.6

7. (B.Frenkin) (8–9) In a non-isosceles triangle ABC the bisectors of angles A and B are inversely proportional to the respective sidelengths. Find angle C.

   Answer. $60°$.

   Solution. Let AA_1, BB_1 be the bisectors of the given triangle, and AA_2, BB_2 be its altitudes. The condition yields that $AA_1/AA_2 = BB_1/BB_2$, therefore, $\angle A_1AA_2 = \angle B_1BB_2$. But $\angle A_1AA_2 = |\angle B - \angle C|$, $\angle B_1BB_2 = |\angle A - \angle C|$. Since the triangle isn't isosceles, an equality $\angle A - \angle C = \angle B - \angle C$ is impossible. Therefore, $\angle C = (\angle A + \angle B)/2 = 60°$.

8. (D.Shvecov) (8–9) Let BM be the median of right-angled triangle ABC ($\angle B = 90°$). The incircle of triangle ABM touches sides AB, AM in points A_1, A_2; points C_1, C_2 are defined similarly. Prove that lines A_1A_2 and C_1C_2 meet on the bisector of angle ABC.

   Solution. Since ABM, CBM are isosceles triangles, points A_1, C_1 are the midpoints of correspondent cathetus. Also the line A_1A_2 is perpendicular to the bisector of angle A, therefore it is the bisector of angle AA_1C_1 (fig.8). Similarly C_1C_2 is the bisector of angle CC_1A_1. Thus its common point is the excenter of triangle A_1BC_1 and lies on the bisector of angle B.

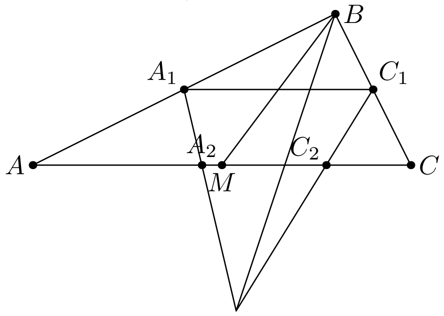

Fig.8

9. (A.Karluchenko) (8–9) In triangle ABC, given lines l_b and l_c containing the bisectors of angles B and C, and the foot L_1 of the bisector of angle A. Restore triangle ABC.

   Solution. Let I be the common point of l_b and l_c. Then IL_1 is the bisector of angle A. Thus we know the angles between the bisectors of the triangle and therefore we know the angles of the triangle. Construct an arbitrary triangle $A'B'C'$ with the same angles, find its incenter I', construct on the lines l_b, l_c the segments $IB'' = I'B'$, $IC'' = I'C'$ and pass the line through L_1 parallel to $B''C''$. This line meets l_b, l_c at the vertices B, C of the sought triangle. The construction of the vertex A is now evident.

10. (B.Frenkin, A.Zaslavsky) In a convex quadrilateral all sidelengths and all angles are pairwise different.

    a)(8–9) Can the greatest angle be adjacent to the greatest side and at the same time the smallest angle be adjacent to the smallest side?

    b)(9–11) Can the greatest angle be non-adjacent to the smallest side and at the same time the smallest angle be non-adjacent to the greatest side?

    Answer. a) Yes. b) No.

    Solution. a) Consider a triangle ABC with $AC > BC > AB$. Take on the segment AC a point P, such that $AP = BC$, construct the perpendicular from P to AC and take on this perpendicular a point D, lying outside the triangle and sufficiently near to P. Then AD is the greatest side of quadrilateral $ABCD$, CD is its smallest side, D is the greatest angle, and C is the smallest angle (fig.10).

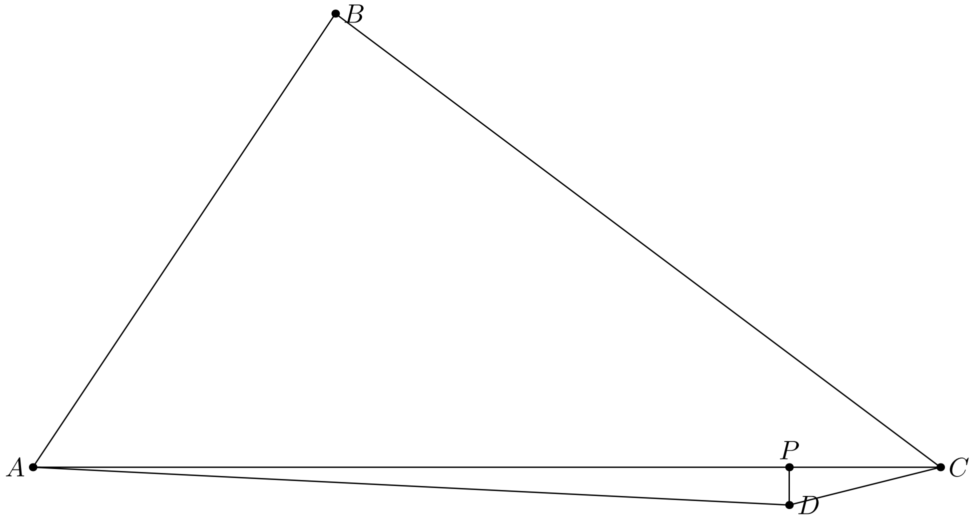

Fig.10

b) Suppose, that $ABCD$ is a quadrilateral satisfying to the condition. We can think that B is the greatest angle, and CD is the smallest side. Then the equality $AC^2 = AB^2 + BC^2 - 2AB \cdot BC \cos B = AD^2 + CD^2 - 2AD \cdot CD \cos D$ yields that AD is the greatest side, therefore, C is the smallest angle. Since $\angle C + \angle D < \pi$, the rays CB and DA meet at some point P. Since angle C is acute and $\angle C + \angle A < \pi$, we obtain that $\sin A > \sin C$. Since $PB/\sin A = AB/\sin P > CD/\sin P = PD/\sin C$, this yields that $PB > PD$. But $PB = PC - BC < PC - CD < PD$ — contradiction.

11. (Tran Q.H.) Given triangle ABC and point P. Points A', B', C' are the projections of P to BC, CA, AB. A line passing through P and parallel to AB meets the circumcircle of triangle $PA'B'$ for the second time in point C_1. Points A_1, B_1 are defined similarly. Prove that

a) (8-10) lines AA_1, BB_1, CC_1 concur;

b) (9–11) triangles ABC and $A_1B_1C_1$ are similar.

Solution. Since PC is the diameter of the circumcircle of $PA'B'$, therefore the angle PC_1C is right, i.e. C_1 lies on the altitude of ABC. Similarly A_1, B_1 lie on the two remaining altitudes. Thus the lines AA_1, BB_1, CC_1 meet on the orthocenter H and the assertion a) is proved. Also A_1, B_1, C_1 lie on the circle with diameter PH, because the angles PA_1H, PB_1H, PC_1H are right. Therefore, the angle between the lines A_1C_1 and B_1C_1 is equal to the angle between the lines HA_1 and HB_1, which as the angle between two altitude of the triangle ABC is equal two the angle between its sidelines AC and BC. Thus the angles of the triangles ABC and $A_1B_1C_1$ are equal, i.e. these triangles are similar.

12. (M.Zhanbulatuly) (9–10) Let O be the circumcenter of an acute-angled triangle ABC. A line passing through O and parallel to BC meets AB and AC in points P and Q respectively. The sum of distances from O to AB and AC is equal to OA. Prove that $PB + QC = PQ$.

Solution. An equality $\cos A + \cos B + \cos C = 1 + r/R$ yields, that in an acute-angled triangle the sum of distances from O to the sides is equal to the sum of the circumradius and the inradius. Thus we obtain that PQ passes through the incenter I. Then $\angle PIB = \angle IBA = \angle IBP$ and $PB = IP$. Similarly $QC = IQ$.

13. (A.Zaslavsky) (9–10) Points A, B are given. Find the locus of points C such that C, the midpoints of AC, BC and the centroid of triangle ABC are concyclic.

Answer. A circle having the center at the midpoint of AB and the radius equal to $AB\sqrt{3}/2$ without its common points with line AB.

Solution. Let the medians AA_0 and BB_0 of the triangle meet at the point M. From the condition we have that $AM \cdot AA_0 = AB_0 \cdot AC$, i.e. $AA_0^2 = \frac{3}{4}AC^2$. Similarly, $BB_0^2 = \frac{3}{4}BC^2$. Since for an arbitrary triangle the ratio of the sums of the squares of its medians and its sides is equal to 3/4, these equalities yield that the median from C is equal to $AB\sqrt{3}/2$. It is clear that all points of the circle distinct from its common points with line AB lie on the sought locus.

14. (M.Volchkevich) (9–10) In a convex quadrilateral $ABCD$ suppose $AC \cap BD = O$ and M is the midpoint of BC. Let $MO \cap AD = E$. Prove that $\frac{AE}{ED} = \frac{S_{\triangle ABO}}{S_{\triangle CDO}}$.

    Solution. Let P be the common point of AB and MO. Applying the Menelaos theorem to triangles ABC and ABD, we obtain that $\frac{AP}{PB} \cdot \frac{BO}{OD} \cdot \frac{DE}{AE} = \frac{AP}{PB} \cdot \frac{BM}{MC} \cdot \frac{CO}{OA} = 1$. Therefore, $\frac{AE}{ED} = \frac{OA \cdot OB}{OC \cdot OD} = \frac{S_{\triangle ABO}}{S_{\triangle CDO}}$.

15. (A.Zaslavsky) (9–11) Given triangle ABC. Consider lines l with the next property: the reflections of l in the sidelines of the triangle concur. Prove that all these lines have a common point.

    Solution. Let the reflections of l concur at the point P. Then the reflections of P lie on l, therefore, the projections of P to the sidelines are collinear. By Simson theorem P lies on the circumcircle of ABC. Since the Simson's line of P bisects the segment between P and the orthocenter H of ABC, we obtain that l passes through H.

16. (F.Ivlev) (9–11) Given right-angled triangle ABC with hypothenuse AB. Let M be the midpoint of AB and O be the center of circumcircle ω of triangle CMB. Line AC meets ω for the second time in point K. Segment KO meets the circumcircle of triangle ABC in point L. Prove that segments AL and KM meet on the circumcircle of triangle ACM.

    First solution. Since $BMKC$ is a cyclic quadrilateral, therefore $\angle BMK = 90°$ and O lies on BK. Thus $\angle ABL = \angle MBK = \angle MCK = \angle A$. Therefore, $\angle MAL = \angle B$, and the angles between AL and KM is equal to angle A, i.e. angle ACM (fig.16).

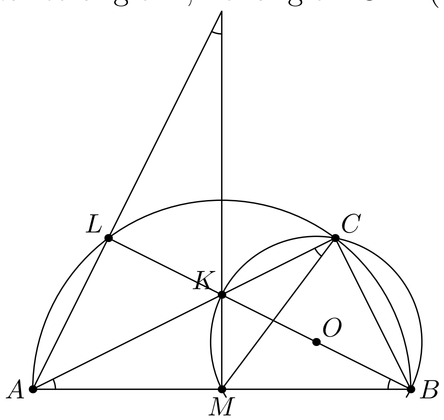

Fig.16

Second solution. Since KCB is a right angle, therefore O lies on KB. Since AB is a diameter of the circumcircle of ABC, therefore ALB is also a right angle. The angle KMB is right, because KCB is a right angle. Thus $K$is the orthocenter of the triangle formed by A, B and the common point of AL and MK. Then two right angles with vertices C and M leans on the same diameter.

17. (M.Rozhkova) (9–11) A square $ABCD$ is inscribed into a circle. Point M lies on arc BC, AM meets BD in point P, DM meets AC in point Q. Prove that the area of quadrilateral $APQD$ is equal to the half of the area of the square.

Solution. Since $\angle AMD = 45° = \angle OAD = \angle ODA$, therefore $\angle AQD = \angle AMD + \angle MAQ = \angle PAD$. Similarly, $\angle APD = \angle ADQ$ (fig.17). Thus the triangles APD and QDA are similar, i.e. $AQ \cdot PD = AD^2$, which yields the assertion of the problem.

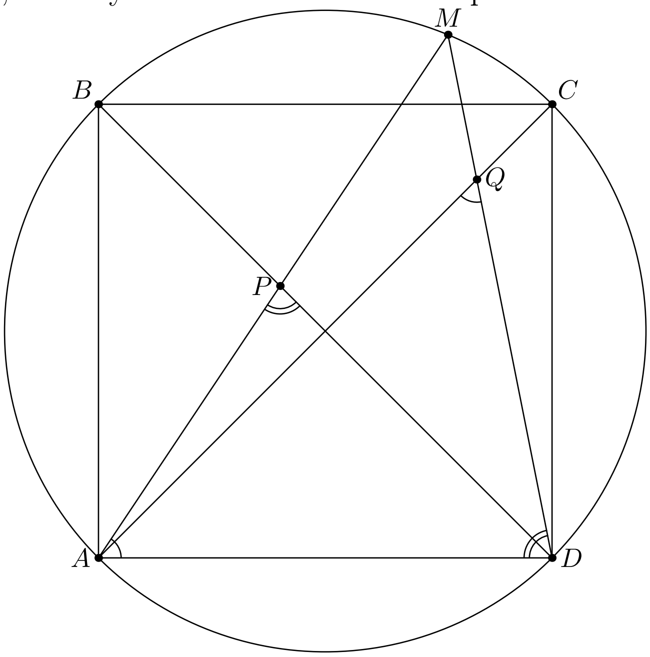

Fig.17

18. (B.Frenkin) (9–11) A triangle and two points inside it are marked. It is known that one of the triangle's angles is equal to 58°, one of two remaining angles is equal to 59°, one of two given points is the incenter of the triangle and the second one is its circumcenter. Using only the ruler without partitions determine where is each of the angles and where is each of the centers.

Solution. Construct the line passing through the marked points. It meets two sides of the triangle (for example AB and AC) and the prolongation of the third side (for example beyond the vertex C). Then AB is the greatest side of the triangle, BC is the smallest side and the marked poind nearest to BC is the incenter.

Prove these assertions. Let I be the incenter of given triangle and O be its circumcenter. Joining them with the vertices of the triangle and calculating the angles we obtain that O lies inside the triangle formed by I and the greatest side , and I lies inside the triangle formed by the smallest side and O. Thus the line OI meets the greatest and the smallest sides, therefore this line meet the prolongation of the third side. Also we obtain that O lies nearer to the greatest side, and I lies neare to the smallest side.

Now we have to examine wyich prolongation of side AC does OI meet. For this compare the lengths of the perpendiculars from O and I to AC. If r is the inradius, and R is the circumradius, then the distance from I to AC is equal to r, and the distance from O to AC is equal to $R\cos\ 59° > R/2 > r$, which yields the answer.

19. (A.Zaslavsky) (10–11) Two circles with radii 1 meet in points X, Y, and the distance between these points also is equal to 1. Point C lies on the first circle, and lines CA, CB are tangents

to the second one. These tangents meet the first circle for the second time in points B', A'. Lines AA' and BB' meet in point Z. Find angle XZY.

Answer. $150°$.

Solution. The condition yields that the distance between the centers of the circles is equal to $\sqrt{3}$, therefore by Euler formula these circles are the circumcircle and the excircle of the triangle $A'B'C$, i.e. $A'B'$ touches the second circle in a point C', lying on the line CZ (fig.19).

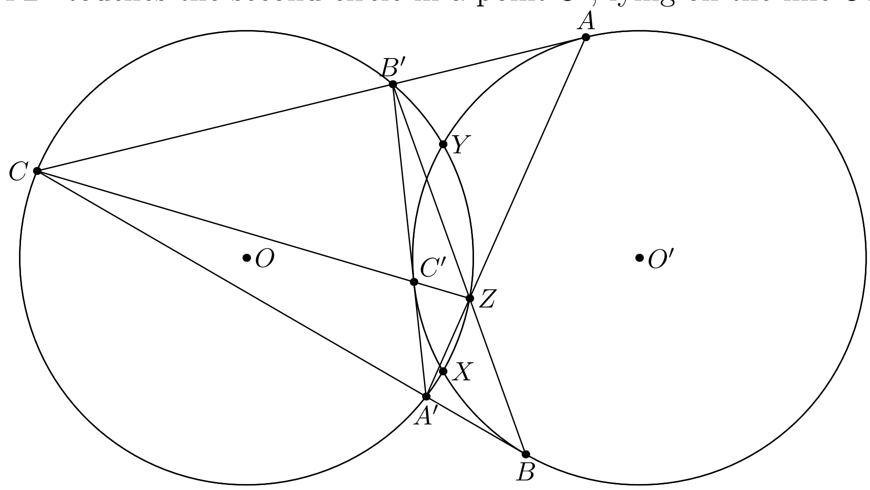

Fig.19

Let O and O' be the centers of the circles. Then $\angle A'O'A = \angle AO'C' + \frac{1}{2}\angle C'O'B = 2\angle ABC' + \angle C'AB = \angle CB'A' + \frac{1}{2}\angle CA'B'$, $\angle O'A'O = \angle O'A'B' + \angle B'A'O = \frac{\pi}{2} - \angle C'O'A' + \frac{\pi}{2} - \angle BCA = \pi - \angle BCA - \frac{1}{2}\angle CA'B' = \angle CB'A' + \frac{1}{2}\angle CA'B'$, and, since $O'A = OA'$, therefore $AO'A'O$ is an isosceles trapezoid. Thus $\angle O'AA' = \angle A'OO'$ and, similarly, $\angle O'BB' = \angle B'OO'$. Therefore, $\angle A'ZB' = 2\pi - \angle AO'B - \angle A'OB' = \pi - \angle C$, i.e. Z lies on the circumcircle and $\angle XZY = 150°$.

Note. We can prove that Z lies on the circumcircle on the other way. The point isogonally conjugated to Z wrt $A'B'C$ is the homothety center of the circles, which is an infinite point because the radii are equal.

20. (G.Feldman) (10–11) Point D lies on side AB of triangle ABC. Let ω_1 and Ω_1, ω_2 and Ω_2 be the incirles and the excircles (touching segment AB) of triangles ACD and BCD. Prove that the common external tangents to ω_1 and ω_2, Ω_1 and Ω_2 meet on AB.

    First solution. Let I_1, J_1, I_2, J_2 be the centers of ω_1, Ω_1, ω_2, Ω_2, and K_1, K_2 be the intersection points of I_1J_1, I_2J_2 with AB (fig.20). Then $I_1K_1/I_1C = J_1K_1/J_1C$, $I_2K_2/I_2C = J_2K_2/J_2C$ and, applying the Menelaos theorem to the triangle CK_1K_2, we obtain that I_1I_2 and J_1J_2 meet AB at the same point. The common external tangents also pass through this point.

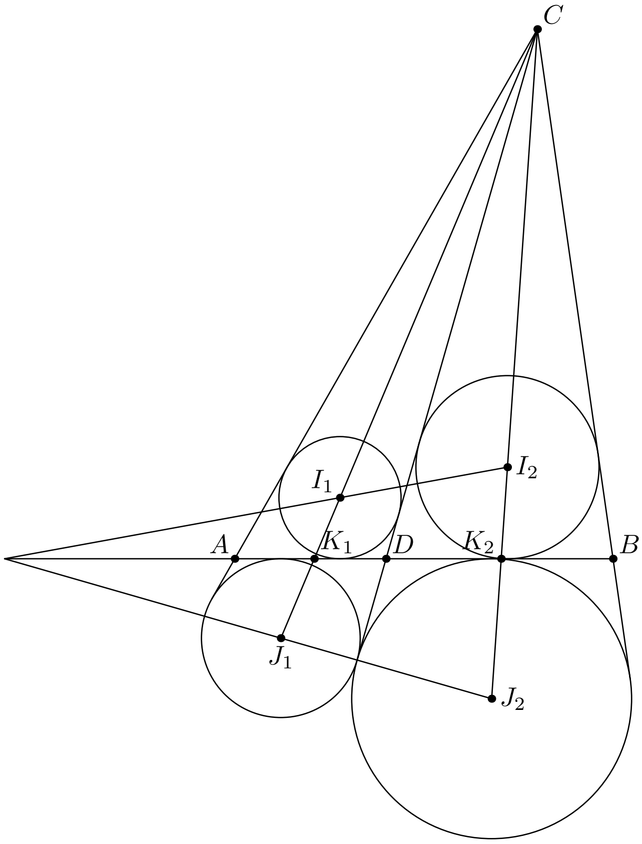

Fig.20

Second solution. Let the common external tangents to ω_1 and Ω_2 meet at a point P. Then applying the three caps theorem to ω_1, Ω_1, Ω_2 and to ω_1, ω_2, Ω_2, we obtain, that the intersection points of the common external tangents to Ω_1, Ω_2 and to ω_1, ω_2 coincide with the common point of the lines PC and AB. Thus these point coincide and lie on AB.

21. (N.Beluhov, E.Colev) (10–11) Two perpendicular lines pass through the orthocenter of an acute-angled triangle. The sidelines of the triangle cut on each of these lines two segments: one lying inside the triangle and another one lying outside it. Prove that the product of two internal segments is equal to the product of two external segments.

Solution. Let one of two lines meets BC, CA, AB at the points X_a, X_b, X_c, and the remaining line meets them at the points Y_a, Y_b, Y_c (fig.21). Then $\angle HY_aB = \angle X_bHA$ and $\angle HX_bA = \angle Y_aHB$, because the sidelines of these angles are perpendicular. Thus the triangles HBY_a and X_bAH are similar. The triangles HX_aB and Y_bAH are also similar. Therefore, $AX_b \cdot BY_a = AH \cdot BH = AY_b \cdot BX_a$. On the other hand applying the Menelaos theorem to the triangles CX_aX_b, CY_aY_b and the line AB, we obtain that $\frac{CA}{AX_b} \cdot \frac{X_bX_c}{X_cX_a} \cdot \frac{X_aB}{BC} = \frac{CA}{AY_b} \cdot \frac{Y_bY_c}{Y_cY_a} \cdot \frac{Y_aB}{BC} = 1$. These three equalities yield the assertion of the problem.

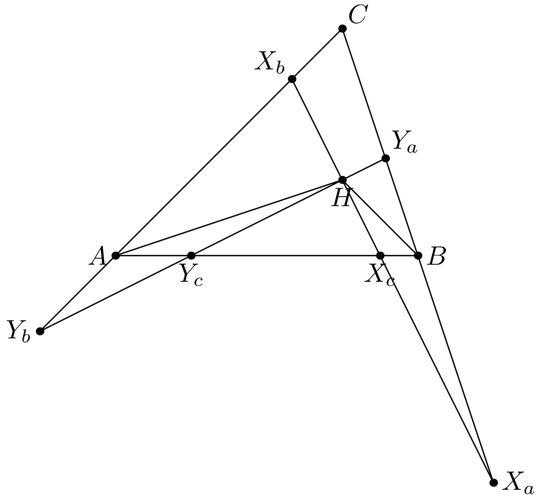

Fig.21

22. (F.Nilov) (10–11) A circle ω with center I is inscribed into a segment of the disk, formed by an arc and a chord AB. Point M is the midpoint of this arc AB, and point N is the midpoint of the complementary arc. The tangents from N touch ω in points C and D. The opposite sidelines AC and BD of quadrilateral $ABCD$ meet in point X, and the diagonals of $ABCD$ meet in point Y. Prove that points X, Y, I and M are collinear.

    Solution. Let K, L — be the touching points of ω with AB and the great circle. Since L is the homothety center of the circles, and the tangents at the points K and N are parallel, therefore the points L, K, N are collinear. Also we have $\angle KAN = \angle NLA$, because the correspondent arcs are equal. Thus the triangles KAN and ALN are similar and $AN^2 = NK \cdot NL = NC^2$, i.e. quadrilateral $ABCD$ is inscribed into a circle with center N (fig.22). The line XY is the polar of the common point of AB and CD wrt this circle. And since $\angle NAM = \angle NBM = \angle NCI = \angle NDI = 90°$, therefore the points M and I are the poles of AB and CD. Thus they lie on XY.

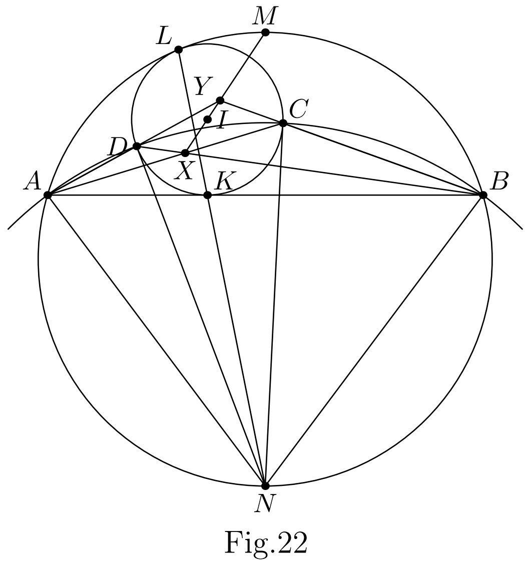

Fig.22

23. (A.Kanel) (10–11) An arbitrary point is selected on each of twelve diagonals of the faces of a cube. The centroid of these twelve points is determined. Find the locus of all these centroids.

    Solution. Firstly note, that the locus of the midpoints of the segments with endpoints lying on two diagonals of a square is the square with the vertices coinciding with the midpoints of the sides of the original square. Thus the locus of the centroids of four points lying on the diagonals of two opposite faces of a cube is the square with the vertices coinciding with the centers of four remaining faces. Therefore we have to find the locus of centroids of three points each of them lies inside one of three such squares. It is clear that all such centroids lie inside an octahedron formed by the centers of the faces of the cube. Also, if one of three points lies on the central plane of the octahedron, and the distances from two remaining points to this plane don't exceed a half of the edge of the cube, then the distance from the centroid to this plane isn't greater than one third of the edge. Therefore all centroids lie inside the polyhedron obtained by the cutting off the octahedron six pyramids with the edges equal to one third of the edge of the octahedron. On the other hand all vertices of this polyhedron and therefore all its points lie on the sought locus.

24. (V.Yassinsky) (10–11) Given are n ($n > 2$) points on the plane such that no three of them aren't collinear. In how many ways this set of points can be divided into two non-empty subsets with non-intersecting convex envelops?

Answer. $n(n-1)/2$.

Solution. Since the convex envelops don't intersect, the two subsets lit on different sides from some line. Thus we have to examine in how many ways the given set of the points can be divided into two subsets by a line. Take a point O of the plane, which don't lie on any line joining the given points, and consider the polar correspondence with center O. The given points correspond to n lines, such that no two of them aren't parallel and no three don't concur. It is easy to prove by induction that these lines divide the plane into $n(n+1)/2+1$ parts, and $2n$ from these parts aren't limited.

Lemma. Let the polars a, b of the points A, B divide the plane into 4 angles. Then the poles of the lines, intersecting the segment AB, lie inside two vertical angles, and the poles of the lines which don't intersecting the segment AB lie inside two remaining angles.

In fact let the lines l and AB meet at the point X. Then the polar of X passes through the common point of a and b. When l rotates around X, its pole moves on this line, i.e. inside some pair of vertical angles formed by a and b. When X moves on AB its polar rotates around the common point of a and b, passing from one pair of vertical angles into the other when X passes through A, B. Lemma is proved.

Return to the problem. The lemma yields that two lines divide the given set of the points by the same way iff their poles lie inside the same part formed by the polars of the given points, or these poles lie on the different sides from all n polars. But the second case is possible iff the two points lie inside the not limited parts. In fact if two points P, Q lie on the different sides from all lines, then each of these lines intersect the segment PQ. Thus each of two rays prolongating this segment lies entirely inside one of the parts. Inversely, if the part containing the point P isn't limined, then ther exists a ray with endpoint in P, lying entirely inside this part and not parallel to any of n lines. The opposite ray intersect all lines and therefore contains a points lying on the different sides than P from these lines.

Thus, $2n$ not limited parts forms n pairs, each of them correspond to one way of dividing of the given set of the points. Each of limited parts also correspond to one way of dividing. Therefore we have $n(n-1)/2+1$ ways, but for one of them all n points belong to the same subset.