XIII GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN
The correspondence round. Solutions

1. (A.Zaslavsky) (8) Mark on a cellular paper four nodes forming a convex quadrilateral with the sidelengths equal to four different primes.

   Solution. Take for example a quadrilateral with vertices $A(-3,0)$, $B(0,4)$, $C(12,-1)$, $D(12,-8)$. Its sidelengths are $AB=5$, $BC=13$, $CD=7$, $DA=17$.

2. (L.Shteingarts) (8) A circle cuts off four right-angled triangles from rectangle $ABCD$. Let A_0, B_0, C_0 and D_0 be the midpoints of the correspondent hypotenuses. Prove that $A_0C_0=B_0D_0$.

   Solution. Let the circle meet AB, BC, CD, DA at points K_1, K_2, L_1, L_2, M_1, M_2, N_1, N_2. Then $K_1K_2M_2M_1$ is an isosceles trapezoid, i.e. $AK_1-DM_1=BK_2-CM_2$, or $AK_1+CM_2=BK_2+DM_1$. Hence the projections of segments A_0C_0 and B_0D_0 to AB, equal to $AB-(AK_1+CM_2)/2$ and $AB-(BK_2+DM_1)/2$ respectively, are congruent. Similarly their projections to BC are congruent, therefore the lengths of these segments are equal.

3. (M.Plotnikov) (8) Let I be the incenter of triangle ABC; H_B, H_C the orthocenters of triangles ACI and ABI respectively; K the touching point of the incircle with the side BC. Prove that H_B, H_C and K are collinear.

   Solution. Since BH_B and CH_C are perpendicular to AI, the quadrilateral BH_BCH_C is a trapezoid and its diagonals divide each other as $BH_B : CH_C$. Since the projections M, N of H_B, H_C to AB and AC respectively coincide with the projections of I to these lines, we obtain that $BM=BK$ and $CN=CK$. Also since $\angle H_BBM=\angle H_CCN=90^\circ-\angle A/2$, the right-angled triangles H_BBM and H_CCN are similar. Therefore $BH_B : CH_C = BK : CK$, and the diagonals of the trapezoid meet at K (fig. 3).

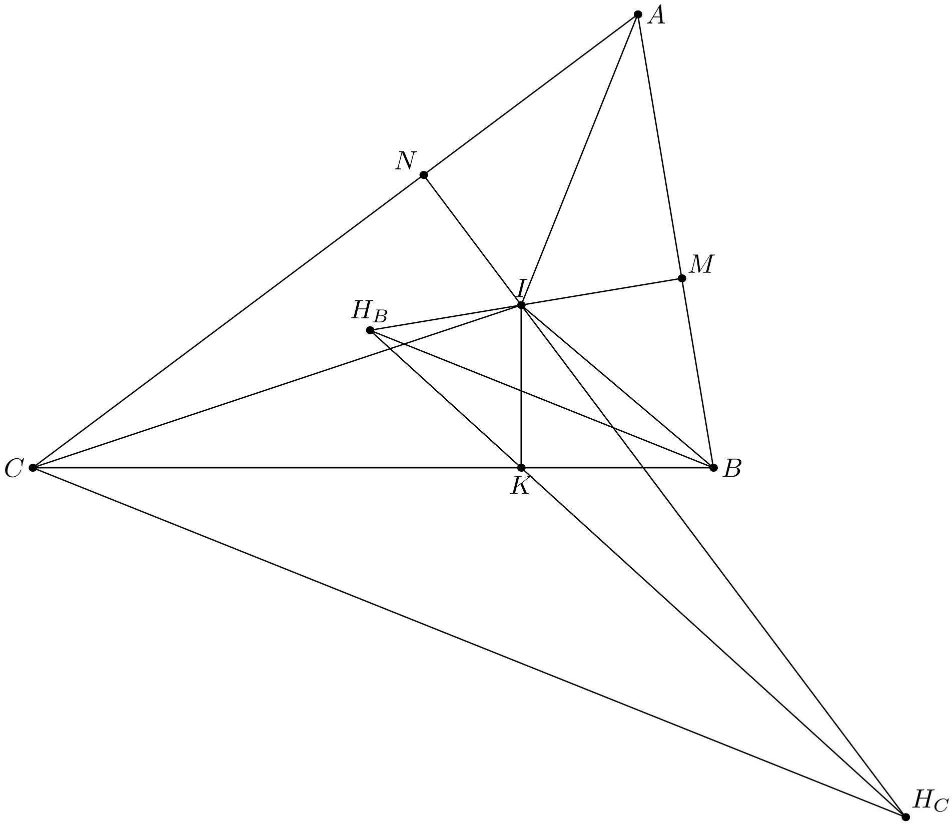

Fig. 3

4. (A.Zaslavsky) (8) A triangle ABC is given. Let C' be the vertex of an isosceles triangle ABC' with $\angle C' = 120°$ constructed on the other side of AB than C, and B' be the vertex of an equilateral triangle ACB' constructed on the same side of AC as ABC. Let K be the midpoint of BB'. Find the angles of triangle KCC'.

   Answer. 90°, 30°, 60°.

   Solution. Let C'' be a vertex of parallelogram $B'C'BC''$. Then $B'C'' = BC' = AC'$, $B'C = AC$ and $\angle CB'C'' = \angle CAC'$ because the angle between $C''B'$ and AC' is equal to $\angle B'CA = 60°$. Therefore the triangles $C''B'C$ and $C'AC$ are congruent, and the angle between their corresponding sidelines $C''C$ and $C'C$ is equal to 60° (fig. 4). Thus the triangle $CC'C''$ is regular, and since K is the midpoint of $C'C''$, we obtain that $CK \perp C'K$ and $\angle C'CK = 30°$.

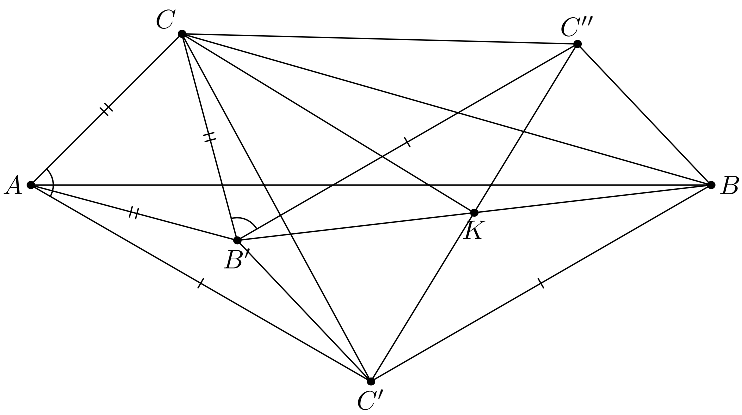

Fig. 4

This reasoning can be also formulated in the following way. Consider the rotations around C' by $120°$ and around C by $60°$. Their composition maps B to B', hence it is the reflection about K. Since it maps C to C'', we obtain the indicated answer.

5. (B.Frenkin) A segment AB is fixed on the plane. Consider all acute-angled triangles with side AB. Find the locus of

   a) (8) the vertices of their greatest angles;

   b) (8–9) their incenters.

   Answer. a) The points A, B and the set of points lying inside or on the boundary of the intersection of two discs centered at A and B with radii AB, but outside the disc with diameter AB. b) The set of points lying inside the square $AKBL$, but outside the intersection of two discs centered at K and L with radii KA.

   Solution. a) If the vertex of the greatest angle does not coincide with A or B then AB is the greatest side of triangle ABC, i.e. $CA \leq AB$ and $CB \leq AB$. On the other hand, since angle C is acute, we obtain that C lies outside the circle with diameter AB.

   b) Let I be the incenter of ABC. Since angles A and B are acute, we have $\angle IAB < 45°$ and $\angle IBA < 45°$, i.e. I lies inside the square $AKBL$. On the other hand, since angle C is acute, we obtain that $\angle AIB < 135°$ and I lies outside the intersection of the discs centered at K, L with radii KA.

6. (N.Moskvitin) (8–9) Let $ABCD$ be a convex quadrilateral with $AC = BD = AD$; E and F the midpoints of AB and CD respectively; O the common point of the diagonals. Prove that EF passes through the touching points of the incircle of triangle AOD with AO and OD.

   Solution. Let X, Y, Z be the touching points of the incircle with AO, OD, AD respectively. Then $DY = DZ$ and therefore $BY = AZ = AX$. Furthermore $OX = OY$. Applying the Menelaus theorem to the triangle AOB and the line XY, we obtain that this line passes through E. Similarly it passes through F (fig. 6).

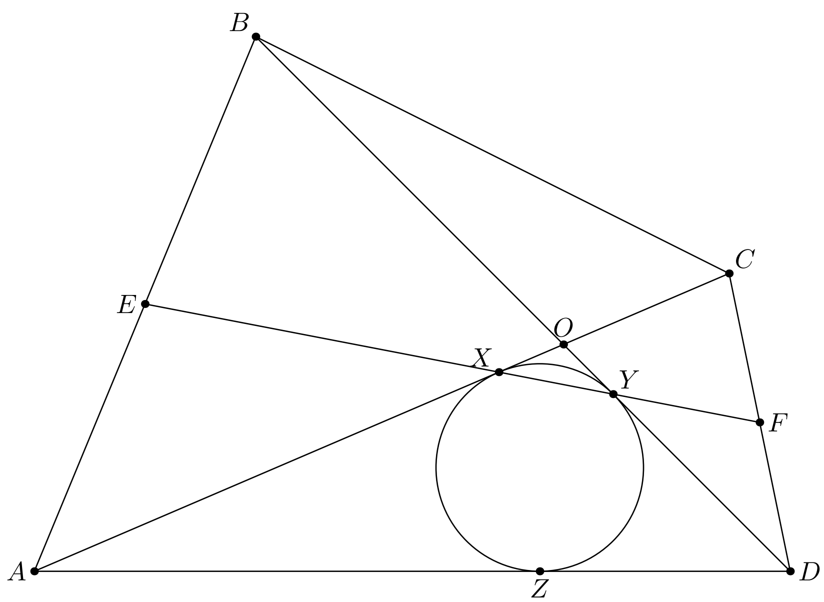

Fig. 6

7. (B.Frenkin) (8–9) The circumcenter of a triangle lies on its incircle. Prove that the ratio of its greatest and smallest sides is less than two.

First solution. Since the circumcenter belongs to the given triangle ABC, this triangle is not obtuse-angled. If it is right-angled then the circumcenter O is the midpoint of its hypothenuse and coincides with the touching point of the incircle. Therefore the triangle is isosceles and right-angled and the assertion of the problem is valid. Suppose that the triangle is acute-angled and O lies on one of three arcs between the touching points. Let this arc is faced to the vertex A. Construct the perpendiculars from O to AB and AC. The foot of each of them (the midpoint of the corresponding side) lies between A and the touching point of the incircle ω with the side. Therefore, $AB > BC$ and $AC > BC$.

Now we have to prove that the ratio of each of sides AB, AC to BC is less than 2. For example let D be the midpoint of AB. Let us prove that $AD < BC$. Let K and L be the touching points of ω with AB and BC. Then $BK = BL$, and we have to prove that $DK < CL$. But the perpendicular from D to AB passes through the point O on ω, hence DK is not greater than its radius. On the other hand CL is greater than the radius, because the perpendicular from C to BC does not intersect ω (the angle between BC and the tangent CA is acute). Q.e.d.

Second solution. Use the Euler formula: $OI^2 = R^2 - 2Rr$, where I is the incenter, R, r are the radii of the circumcircle and the incircle. Since $OI = r$ we obtain that $r/R = \sqrt{2} - 1$. Each side of the triangle is a chord of the circumcircle tangent to the incircle. The greatest of these chords is equal to $2R$, and the shortest one touching the incircle at the point opposite to O is $2\sqrt{R^2 - 4r^2} > R$.

8. (Ye.Bakayev) (8–9) Let AD be the base of trapezoid $ABCD$. It is known that the circumcenter of triangle ABC lies on BD. Prove that the circumcenter of triangle ABD lies on AC.

Solution. Let the perpendicular bisector to AB meet BD and AC at points K and L respectively. Then by the assumption $\angle BLK = \angle ACB = \angle CAD$. Hence $\angle CKL = \angle BDA$ which yields the required assertion (fig. 8).

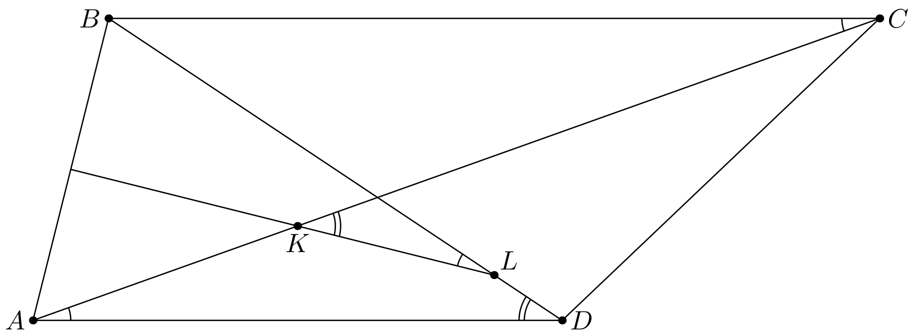

Fig. 8

9. (A.Zaslavsky) (8–9) Let C_0 be the midpoint of hypotenuse AB of triangle ABC; AA_1, BB_1 the bisectors of this triangle; I its incenter. Prove that the lines C_0I and A_1B_1 meet on the altitude from C.

Solution. Use the following property of an arbitrary triangle.

Lemma. The line C_0I meets the altitude CH at the point lying at the distance r from C.

In fact, let C', C'' be the touching points of side AB with the incircle and the excircle respectively, and C_2 the point of the incircle opposite to C'. Point C is the homothety center of the incircle and the excircle, and C_2 and C'' are the corresponding points of these circles, therefore C, C_2, C'' are collinear. Furthermore $C'C_0 = C''C_0$, i.e. C_0I is the medial line of triangle $C'C''C_2$, and $C_0I \parallel CC_2$. Hence the lines CC_2, C_2I, C_0I and CH are the sidelines of a parallelogram, and we obtain the assertion of the lemma (fig. 9.1).

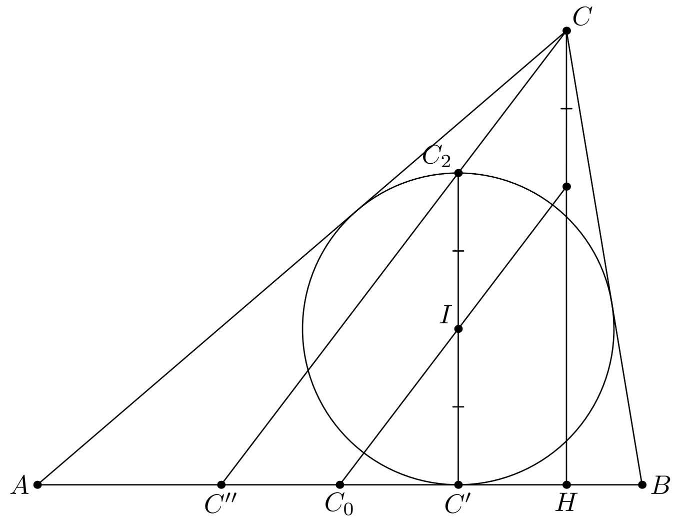

Fig. 9.1

Return to the problem. Denote the common point of C_0I and CH by H' (fig. 9.2). Since $CH' = r$, the distances from H' to CA, BC and AB are $d_b = r\cos\angle HCB = r\cos\angle BAC = r\cdot AC/AB$, $d_a = r\cdot BC/AB$ and $d_c = CH - r$ respectively. Since $(AB + BC + CA)r = AB\cdot CH = 2S_{ABC}$, we obtain that $d_c = d_a + d_b$. It is clear that the distances from A_1, B_1 to BC, CA andAB also have the similar property. By the Thales theorem all such points are collinear.

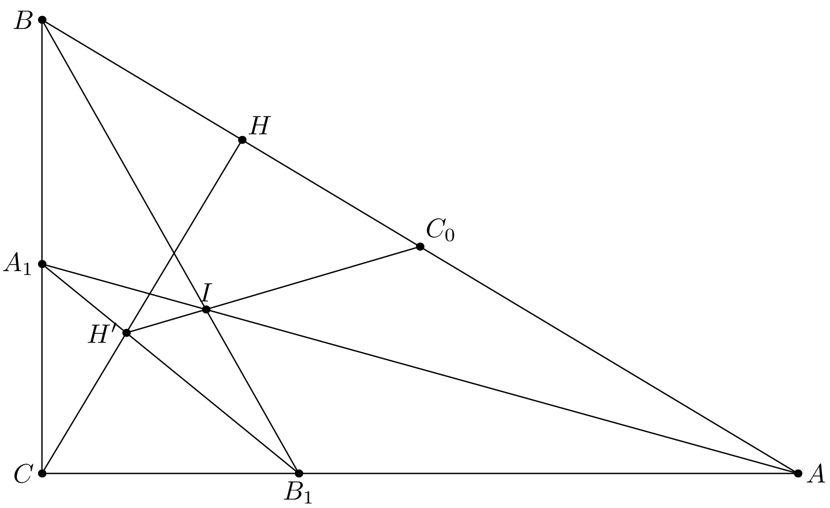

Fig. 9.2

10. (I.I.Bogdanov) (8–10) Points K and L on the sides AB and BC of parallelogram $ABCD$ are such that $\angle AKD = \angle CLD$. Prove that the circumcenter of triangle BKL is equidistant from A and C.

    Solution. The triangles AKD and CLD are similar by two angles, therefore $AK : CL = AD : CD$. Hence, when K moves along AB with constant velocity, L also moves along BC uniformly, and therefore the circumcenter of BKL moves along some line. If K, L are the projections of D to AB and BC respectively, the circumcenter of BKL coincides with the center of the parallelogram, and when K and L coincide with A and C respectively, the circumcenter lies on the perpendicular bisector to AC. Thus this perpendicular bisector is the locus of circumcenters.

11. (A.Tolesnikov) (8–11) A finite number of points is marked on the plane. Each three of them are not collinear. A circle is circumscribed around each triangle with marked vertices. Is it possible that all centers of these circles are also marked?

    Answer. No.

    Solution. Consider the circle having the minimal radius. Let it be the circumcircle of triangle ABC, and O be its center. If ABC is not a regular triangle, then some of its angles, for example C, is less than $60°$. But in this case $60° < \angle AOB < 120°$, i.e. $\sin\angle AOB > \sin\angle ACB$, and by the sinus theorem the circumradius of AOB is less than the radius of circle ABC, which contradicts to the definition of this circle. If ABC is regular then the centers A', B', C' of circles BOC, COA, AOB are also marked. But for example the triangle AOB' is regular, and its circumradius is less than the radius of circle ABC.

12. (D.Shvetsov) (9–10) Let AA_1, CC_1 be the altitudes of triangle ABC, B_0 the common point of the altitude from B and the circumcircle of ABC; and Q the common point of the circumcircles of ABC and $A_1C_1B_0$, distinct from B_0. Prove that BQ is the symmedian of ABC.

    Solution. Since A, C, A_1, C_1 are concyclic we obtain that the lines AC, A_1C_1 and B_0Q concur at the radical center N of circles ACA_1C_1, ABC and $A_1C_1B_0$. Let BQ meet AC and A_1C_1 at points P and M respectively (fig. 12). Projecting the circumcircle of triangle ABC from Q to AC, and projecting this line from B to A_1C_1 we obtain te equality of cross-ratios $(A_1C_1MN) = (CAPN) = (CABB_0) = \frac{BC}{BA} : \frac{B_0C}{B_0A}$. Since B_0 is the reflection of the orthocenter H of triangle ABC about AC, the second fraction is equal to $HC/HA = CA_1/AC_1$. Now applying the Menelaus theorem to the triangle A_1BC_1 and the line ACN we obtain that $A_1C_1MN = C_1N/A_1N$, i.e. $A_1M = C_1M$. Therefore BM is the median of triangle A_1BC_1 and the symmedian of ABC.

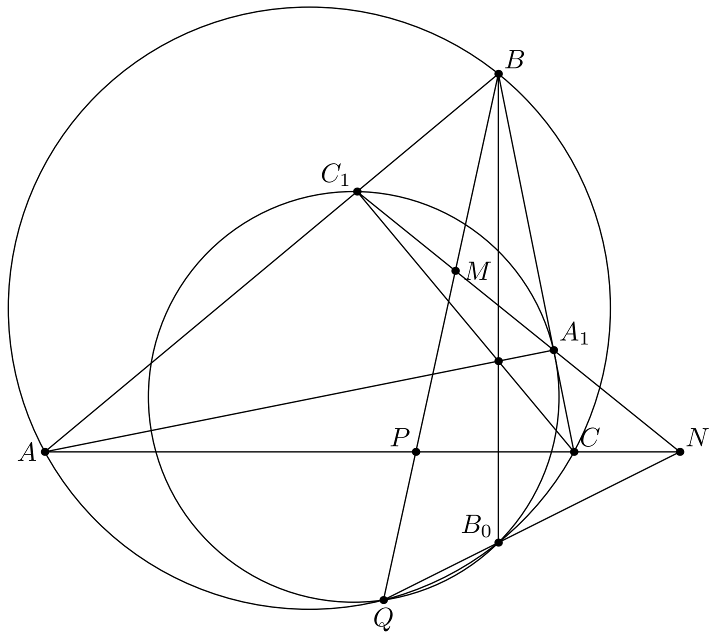

Fig. 12

13. (A.Zaslavsky) (9–11) Two circles pass through points A and B. A third circle touches both these circles and meets AB at points C and D. Prove that the tangents to this circle at these points are parallel to the common tangents of two given circles.

Solution. Let the third circle touch two given circles at points X, Y, and their common tangent touch them at U, V (points X and U lie on the same circle). Since X is the homothety center of touching circles, the line XU meets the third circle at point P such that the tangent at this point is parallel to UV. Similarly YV passes through P. Also X, Y, U, V are collinear, therefore $PX \cdot PU = PY \cdot PV$. Hence P lies on AB and thus coincides with one of points C, D (fig. 13). The proof for the second point is similar.

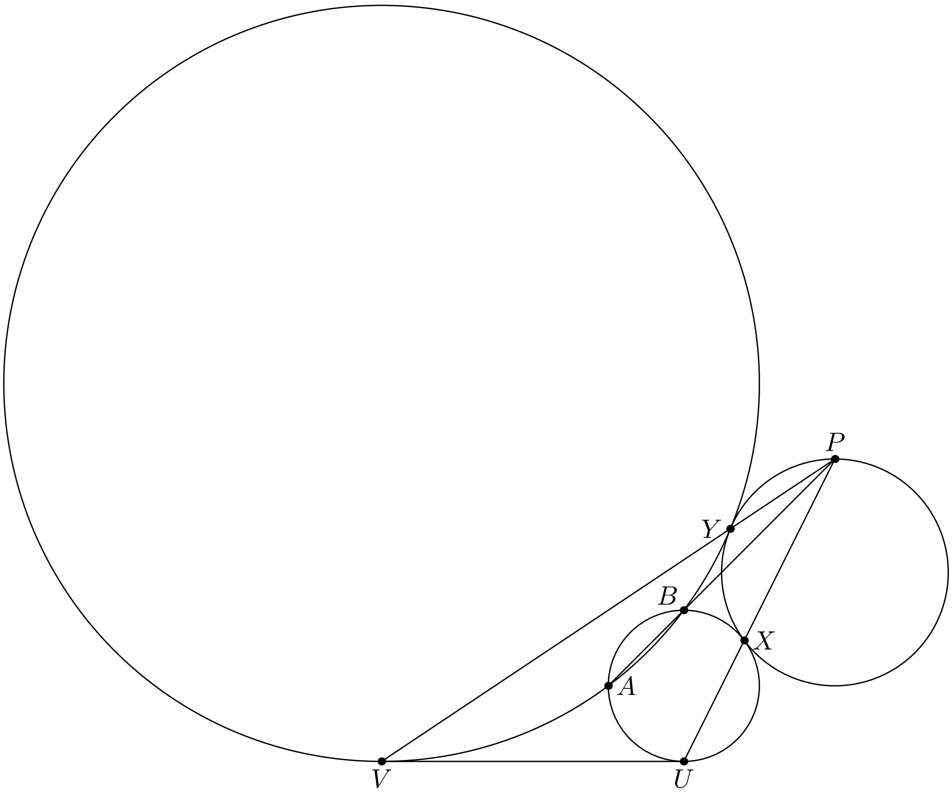

Fig. 13

14. (N.Moskvitin) (9–11) Let points B and C lie on the circle with diameter AD and center O on the same side of AD. The circumcircles of triangles ABO and CDO meet BC at points F and E respectively. Prove that $R^2 = AF \cdot DE$, where R is the radius of the given circle.

Solution. Since $ABFO$ is cyclic and $AO = OB$, we have (fig.14)

$$\frac{AF}{AO} = \frac{\sin\angle AOF}{\sin\angle ABO} = \frac{\sin\angle ABF}{\sin\angle ABO} = \frac{\sin\angle ABC}{\sin\angle BAD}.$$

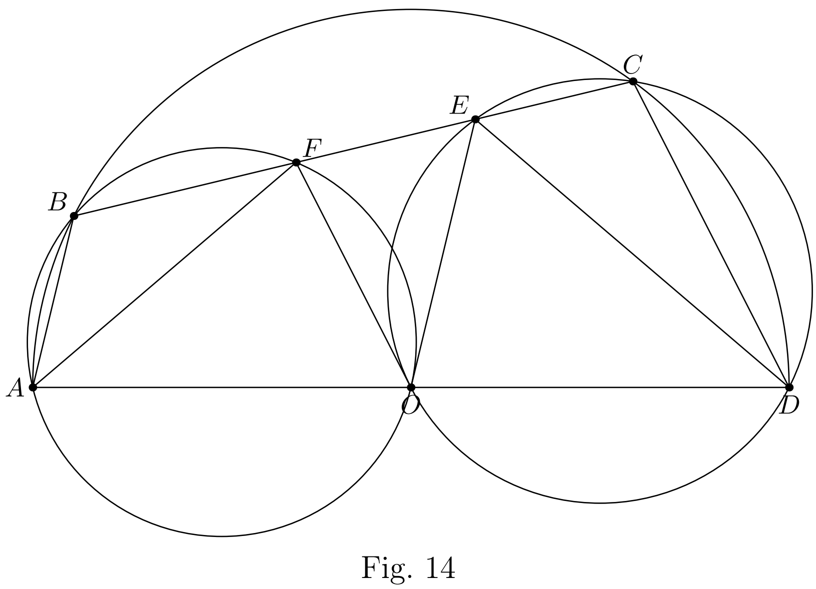

Fig. 14

Similarly, $DE/OD = \sin\angle BCD/\sin\angle CDA$. Since $ABCD$ is cyclic, the product of these ratios is equal to 1.

15. (K.Aleksiev) (9–11) Let ABC be an acute-angled triangle with incircle ω and incenter I. Let ω touch AB, BC and CA at points D, E, F respectively. The circles ω_1 and ω_2 centered at J_1 and J_2 respectively are inscribed into $ADIF$ and $BDIE$. Let J_1J_2 intersect AB at point M. Prove that CD is perpendicular to IM.

    Solution. Since DJ_1, DJ_2 are the bisectors of triangles DIA, DIB respectively, we have $AJ_1/J_1I = AD/ID$, $IJ_2/J_2B = CI/CB$. By the Menelaus theorem we obtain that the quadruple A, B, C, M is harmonic, i.e. M lies on FE (fig.15). Since C and D are the poles of lines EF and AB wrt the incircle we obtain that M is the pole of CD, therefore $CD \perp IM$.

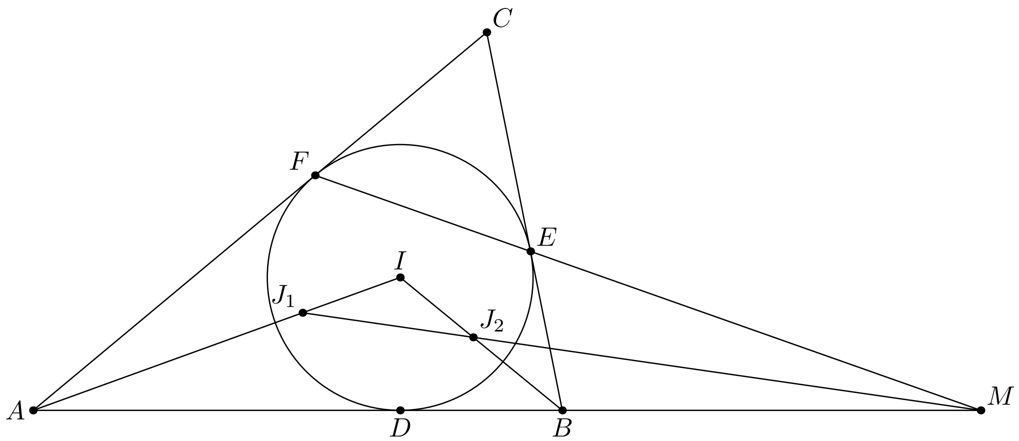

Fig. 15

16. (P.Ryabov) (9–11) The tangents to the circumcircle of triangle ABC at A and B meet at point D. The circle passing through the projections of D to BC, CA, AB, meet AB for the second time at point C'. Points A', B' are defined similarly. Prove that AA', BB', CC' concur.

    Solution. The pedal circle of point D coincides with the pedal circle of isogonally conjugated point D' which is the vertex of parallelogram $ACBD'$. Hence C' is the projection of D' to AB, i.e. the reflection of the foot of the altitude from C about the midpoint of AB. Similarly A', B' are the reflections of the feet of the altitudes from A and B about the midpoints of the corresponding sides. Therefore AA', BB' and CC' concur at the point isotomically conjugated to the orthocenter of the triangle.

17. (A.Trigub) (9–11) Using a compass and a ruler, construct a point K inside an acute-angled triangle ABC so that $\angle KBA = 2\angle KAB$ and $\angle KBC = 2\angle KCB$.

    Solution. Let the circle centered at K and passing through B meet AB and BC at points P and Q respectively, and let T be the midpoint of arc ABC of the circumcircle. Then $\angle KPB = \angle KBP = 2\angle KAP$, therefore $\angle KAP = \angle PKA$ and $AP = PK = KB$. Similarly $CQ = QK = KB$. Since $AP = CQ$, $AT = CT$ and $\angle PAT = \angle QCT$, the triangles TAP and TCQ are congruent i.e. $\angle TPB = \angle TQB$ and T lies on the circle BPQ.

Hence the center K of this circle lies on the perpendicular bisector to BT. Furthermore by the assumption $\angle AKC = 3\angle B/2$, i.e. K lies on the corresponding arc (fig.17).

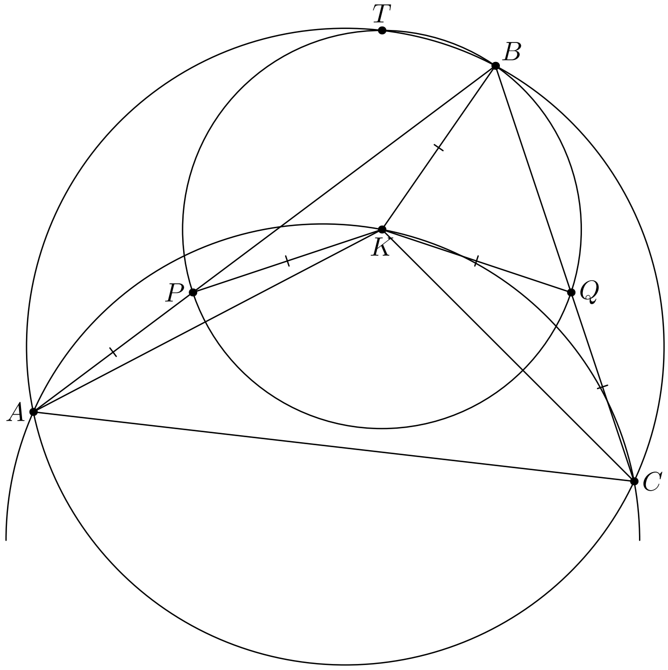

Fig. 17

Now let us prove that the constructed point K is in fact the required one. Denote again the common points of the sidelines with the circle centered at K and passing through B by P and Q. Since this circle passes through T, we obtain that $AP = CQ$. If $AP > PK = KB$ then $\angle PKA > \angle PAK$, $\angle KPB = \angle KBP > 2\angle BAK$, $\angle KBC > 2\angle KCB$ and $\angle AKC < 3\angle B/2$ which contradicts to the construction of K. Similarly if $AP < PK$ we have $\angle AKC > 3\angle B/2$.

18. (A.Trigub) (9–11) Let L be the common point of the symmedians of triangle ABC, and BH be its altitude. It is known that $\angle ALH = 180° - 2\angle A$. Prove that $\angle CLH = 180° - 2\angle C$.

    Solution. Let AA_1, CC_1 be the altitudes of the triangle. Then the symmedians AL, CL are the medians of triangles AC_1H, CA_1H, i.e. they pass through the midpoints M, N of segments HC_1, HA_1 respectively. But $\angle MNH = \angle C_1A_1H = 180° - 2\angle A$, therefore $\angle ALH = 180° - 2\angle A$ if and only if $HLMN$ is cyclic. Similarly this is equivalent to the condition $\angle CLH = 180° - 2\angle C$.

19. (D.Prokopenko) (10–11) Let cevians AA', BB' and CC' of triangle ABC concur at point P. The circumcircle of triangle $PA'B'$ meets AC and BC at points M and N respectively, and the circumcircles of triangles $PC'B'$ and $PA'C'$ meet AC and BC for the second time respectively at points K and L. The line c passes through the midpoints of segments MN and KL. The lines a and b are defined similarly. Prove that a, b and c concur.

    Solution. By the assumption $CM\cdot CB' = CN\cdot CA'$ and $CK\cdot CB' = CP\cdot CC' = CL\cdot CA'$. Hence $KL \parallel MN$ and c passes through C. Since MN and $A'B'$ are antiparallel, this line is the symmedian of triangle $CA'B'$ and so it divides C into two angles with the ratio of sinuses equal to $CB' : CA'$. The similar relations for two remaining angles and the Ceva theorem yield the required assertion.

20. (V.Luchkin, M.Fadin) (10–11) Given a right-angled triangle ABC and two perpendicular lines x and y passing through the vertex A of its right angle. For an arbitrary point X on x define y_B and y_C as the reflections of y about XB and XC respectively. Let Y be the common point of y_b and y_c. Find the locus of Y (when y_b and y_c do not coincide).

    Solution. Consider the point X' isogonally conjugated to X and its reflections U, V, W about AB, AC, BC respectively. Perpendicularity of x and y implies that U and V lie on y. Furthermore XB, XC are the perpendicular bisectors to UW, VW respectively. Therefore W lies on y_b, y_c, i.e. it coincides with Y (fig.20). Thus Y lies on the reflection of the isogonal image of x about BC. The required locus is this line without the points such that y_b and y_c coincide, i.e. the common point of this line with BC and the reflection of A about BC.

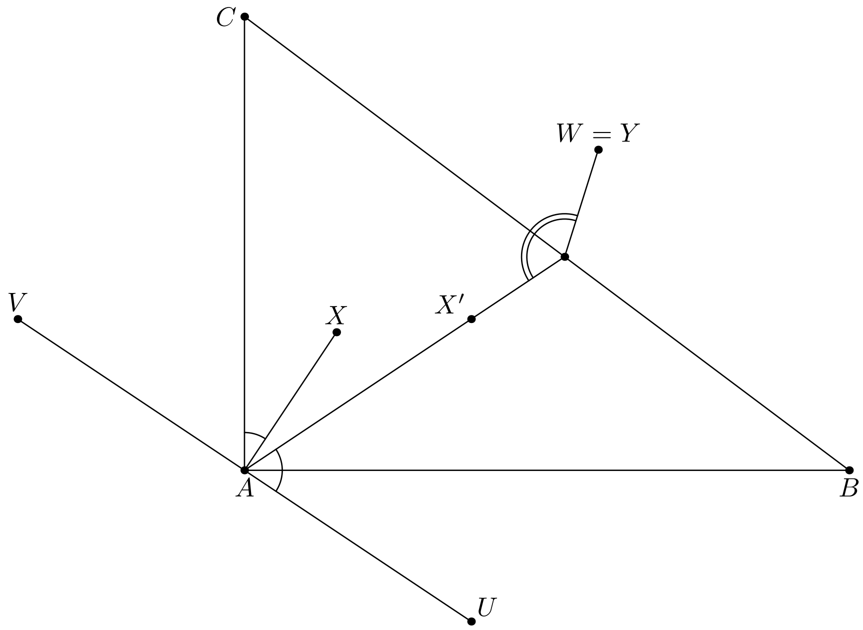

Fig. 20

21. (N.Beluhov) (10–11) A convex hexagon is circumscribed about a circle of radius 1. Consider the three segments joining the midpoints of its opposite sides. Find the greatest real number r such that the length of at least one segment is at least r.

    Solution. Let $A_1A_2\ldots A_6$ be the hexagon in question, circumscribed about a circle ω with center I, and let M_i be the midpoint of A_iA_{i+1} (indices run modulo 6, so that, say, $A_7 \equiv A_1$). If $A_1A_2A_3$ approaches an equilateral triangle and A_4, A_5, and A_6 all approach the midpoint of A_1A_3 then the lengths of M_1M_4, M_2M_5, and M_3M_6 all approach $\sqrt{3}$.

    We will show that $r = \sqrt{3}$ is indeed the answer to the problem. First we verify that I lies inside $M_1M_2\ldots M_6$. Suppose for example that it lies inside the triangle $M_1A_1M_6$. But then ω is contained inside $A_2A_1A_6$ and cannot touch all sides of $A_1A_2\ldots A_6$, a contradiction.

Let $\angle(ABCD)$ denote the angle such that rotation by it counterclockwise about A makes $\overrightarrow{AB}$ codirectional with $\overrightarrow{CD}$.

Since all M_i lie outside ω, we have $IM_i \geq 1$. Therefore, if $120° \leq \angle(IM_iIM_{i+3}) \leq 240°$ for some i then $M_iM_{i+3} \geq \sqrt{3}$ and we are done.

Suppose now that this does not happen for any i. Let j be such that $\angle(IM_jIM_{j+3}) \leq 120°$ and $\angle(IM_{j+3}IM_j) \geq 240°$. Then there is some k, $j \leq k \leq j+2$, such that $\angle(IM_kIM_{k+3}) \leq 120°$ and $\angle(IM_{k+1}IM_{k+4}) \geq 240°$. Without loss of generality, take $k = 4$. Then $120° \leq \angle IM_1IM_2 \leq 180°$ and consequently $M_1M_2 \geq \sqrt{3}$.

Consider the convex quadrilateral $M_1M_2M_4M_5$. If angle M_1 is right or obtuse then $M_2M_5 > M_1M_2 \geq \sqrt{3}$ and we are done. If angle M_2 is right or obtuse then $M_1M_4 > M_1M_2 \geq \sqrt{3}$ and we are done. It remains to consider the case when angles M_1 and M_2 are both acute.

In this case however $90° < \angle(M_1M_2M_4M_5) < 270°$. Since $\overrightarrow{M_3M_6} = -\overrightarrow{M_1M_2} + \overrightarrow{M_4M_5}$ (because $\overrightarrow{M_3M_6} = \overrightarrow{M_3M_4} + \overrightarrow{M_4M_5} + \overrightarrow{M_5M_6}$ and $\overrightarrow{M_1M_2} + \overrightarrow{M_3M_4} + \overrightarrow{M_5M_6} = \frac{1}{2}(\overrightarrow{A_1A_3} + \overrightarrow{A_3A_5} + \overrightarrow{A_5A_1}) = \mathbf{0}$), we have $M_3M_6 > M_1M_2 \geq \sqrt{3}$, and the proof is complete.

22. (M. Panov) (10–11) Let P be an arbitrary point on the diagonal AC of cyclic quadrilateral $ABCD$, and PK, PL, PM, PN, PO be the perpendiculars from P to AB, BC, CD, DA, BD respectively. Prove that the distance from P to KN is equal to the distance from O to ML.

    Solution. When P moves uniformly along AC, the lines KN and ML are translated uniformly and the point O moves uniformly as well. Thus $d(P, KN) - d(O, ML)$ is a linear function of the position of P. When $P = A$, this function equals 0 by the Simson theorem, and when P is the common point of AC and BD, it equals 0 because $KLMN$ is circumscribed about a circle centered at $P = O$ ($\angle NKP = \angle DAC = \angle DBC = \angle PKL$ because $AKPN$ and $BKPL$ are cyclic).

23. (I.Frolov) (10–11) Let a line m touch the incircle of triangle ABC. The lines passing through the incenter I and perpendicular to AI, BI, CI meet m at points A', B', C' respectively. Prove that AA', BB' and CC' concur.

    Solution. The polar transformation wrt the incircle maps BC, CA, AB, m to their touching points A_1, B_1, C_1 M with the incircle. Since IA' is the polar of the infinite point of perpendicular line IA, its common point with m is the pole of the line passing through M and parallel to IA. Since $IA \perp B_1C_1$, the line AA' is the polar of the projection of M to B_1C_1. Similarly the lines BB' and CC' are the polars of projections of M to A_1C_1 and A_1B_1 respectively. By the Simson theorem these projections are collinear, hence their polars concur.

24. (I.I.Bogdanov) (11) Two tetrahedrons are given. Each two faces of the same tetrahedron are not similar, but each face of the first tetrahedron is similar to some face of the second one. Does this yield that these tetrahedrons are similar?

    Answer. No.

    Solution. Let t be some number close to 1. Then there exist two tetrahedrons such that their bases are regular triangles with side equal to 1, the lateral edges of the first tetrahedron are equal to t, t^2, t^3, and the lateral edges of the second one are equal to $1/t$,

$1/t^2$, $1/t^3$. It is clear that the assumption is valid for these tetrahedrons but they are not similar.