Inequalities and derivatives

Theorems

A. If $f'(x)>0 \quad \forall x\in(a,b)$ then $f(x)$ is increasing on $[a,b]$, in particular, $f(a)<f(b)$.

B. If $f''(x)>0 \quad \forall x\in(a,b)$ then $f(x)$ is convex on $[a,b]$, in particular, $f(pa+qb)<p\,f(b)+q\,f(b) \quad \forall p,q\in[0,1]$ such that $p+q=1$.

C. (Jensen's inequality) Let $f(x)$ be convex and numbers $a_{1,}a_{2,}...,a_n$ postive with the sum 1. Then $f(a_1x_1+...+a_nx_n)<a_1f(x_1)+...+a_nf(x_n)$.

1. Prove that $\sin x< x \quad \forall x>0$
2. Prove that $|x+\frac{1}{x}|\geq 2 \quad \forall x\neq 0$
3. Prove that a) $e^x\geq x+1 \quad \forall x$

   b) $e^x>\frac{x^2}{2}+x+1 \quad \forall x>0$ and $e^x<\frac{x^2}{2}+x+1 \quad \forall x<0$

   c) $e^x\geq\frac{x^n}{n!}+...+\frac{x^2}{2}+x+1 \quad \forall x$ and any odd n

4. Compare e^{π} and π^e ?

5. Prove that $\cos\sqrt{x}\geq 1-\frac{x}{2} \quad \forall x\geq 0$
6. Prove that

   a) $\cos\frac{x+y}{2}>\frac{\cos x+\cos y}{2} \qquad \forall x,y\in\left[\frac{-\pi}{2},\frac{\pi}{2}\right]$

   b) $\sqrt{\frac{x+y+z}{3}}>\frac{\sqrt{x}+\sqrt{y}+\sqrt{z}}{3} \qquad \forall x,y,z>0$

   c) $e^{0.3x+0.7y}<0.3e^x+0.7e^y \qquad \forall x,y$

   d) $\tan 50^\circ+\tan 51^\circ+\tan 52^\circ+...+\tan 70^\circ>21\sqrt{3}$

7. Prove that a) $\ln\frac{x_1+x_2+...+x_n}{n}\geq\frac{\ln x_1+\ln x_2+...\ln x_n}{n}$

   b) $\frac{a_1+a_2+...+a_n}{n}\geq\sqrt[n]{a_1a_2...a_n}$

Extra problems

8. Given positive numbers p and q such that $\frac{1}{p}+\frac{1}{q}=1$, prove that $ab\le\frac{a^p}{p}+\frac{b^q}{q}$ for all positive a, b.
9. Prove that $\sin 2x+\cos x>1 \quad \forall x\in(0,\frac{\pi}{3})$
10. (**IMC 2009.2.2**) Let $f:\mathbb{R}\to\mathbb{R}$ be a 2 times differentiable function satisfying
    $f(0)=1, f'(0)=0$ and
    $f''(x)-5f'(x)+6f(x)\ge 0 \quad \forall x\ge 0$
    Prove that
    $f(x)\ge 3e^{2x}-2e^{3x} \quad \forall x\ge 0$
11. (**IMC 2006.2.3**) Compare functions $\sin(\tan x)$ and $\tan(\sin x)$ for all $x\in(0,\frac{\pi}{2})$
12. Polynomials P and Q are quadratic, and inequalities $P(x)\le 0$ and $Q(x)\le 0$ has no common solutions. Prove that there exist positive numbers a and b such that the polynomial $aP+bQ$ is positive for all x.

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