# Polynomials with integer coefficients

**1.** Let $P$ be a polynomials with integer coefficients (in short, $P\in\mathbb{Z}[x]$ ). Prove that $(a-b)|(P(a)-P(b))\quad\forall a,b\in\mathbb{Z}$

**2.** Let $P\in\mathbb{Z}[x]$ . Prove that

**a)** if $a$ and $b$ are integers of the same parity then $P(a)$ and $P(b)$ are of the same parity.

**b)** if $a$ and $b$ are integers such that $P(a) = 0$ and $P(b) = 1$, then either $a = b + 1$, or $b = a + 1$.

**3.** A polynomial $P$ is such that $P(7)=11$ and $P(11)=13$. Prove that at least one of the coefficients of $P$ is not integer.

**4.** Does there exist $P\in\mathbb{Z}[x], P\neq const$ such that $P(x)$ is a prime number for any integer $x$?

**5.** Let $H(x)=a_n x^n+...+a_1 x+a_0$ be the polynomial with integer coefficients and $\frac{p}{q}$ an irreducible fraction such that $H(\frac{p}{q})=0$ . Prove that **a)** $p|a_0$ and **b)** $q|a_n$

**Definition.** Fix a set K of coefficients, e.g., $K=\mathbb{Z},\mathbb{Q},\mathbb{R},\mathbb{Z}_p,\mathbb{C}$ . Let K[x] be the set of polynomials with coefficients in K. A polynomial of K[x] is said to be *irreducible* over K if it does not split into the product of lesser degree polynomials of K[x].

**6.** Over $\mathbb{Q}$ , factorize the polynomials **a)** $x^4+4$ **b)** $x^4+1$ **c)** $x^6-11x^4+36x^2$ into irreducible polynomials.

## *Extra problems*

**PIC1.** On the graph of a polynomial with integer coefficients two points with integer coordinates are marked. Prove that if the distance between them is integer, then the segment connecting them is parallel to the $x$-axis.

**PIC2.** Let $P$ be a nonconstant polynomial with integer coefficients. Prove that among numbers $P(1), P(2), P(3), ...$ there are infinitely many nonprime numbers.

**PIC3.** $P\in\mathbb{Z}[x]$ is irreducible over $\mathbb{Z}$ . Prove that there exist $Q\in\mathbb{Z}[x]$ such that $P(Q)$ is reducible over $\mathbb{Z}$ .

**IMC07.1.1** Let $f$ be a polynomial of degree 2 with integer coefficients. Suppose that $f(k)$ is divisible by 5 for every integer $k$. Prove that all coefficients of $f$ are divisible by 5.

**IMC08.1.3** Let $p$ be a polynomial with integer coefficients and let $a_1<a_2<...<a_k$ be integers.

**a)** Prove that there exists $a\in\mathbb{Z}$ such that $p(a_i)$ divides $p(a)$ for all $i=1,2,...,k$.

**b)** Does there exist $a\in\mathbb{Z}$ such that the product $p(a_1)\cdot p(a_2)\cdot...\cdot p(a_k)$ divides $p(a)$?