Matrices: Convenient Basis

1. Find the matrices of the following linear maps in the given basis
a) $D_n : \mathbb{R}_n[x] \to \mathbb{R}_n[x]$ in the basis $1, x, \ldots, x^n$, where $D_n(P) = P'$
b) i $S_n : \mathbb{R}_n[x] \to \mathbb{R}_n[x]$ n the basis , $1, x, \ldots, x^n$ where $S_n(P(x)) = P(2x-1)$ (for n=2)
c) $R_u : \mathbb{R}_2 \to \mathbb{R}_2$ in the standard basis, where R_u is rotation through an angle u counter-clockwise around the origin.
2. Find the product of the matrices from the previous problem:
a) $D_2 S_2$ **b)** $S_2 D_2$ **c)** $R_a R_b$ **d)** D_3^2 **e)** D_3^3 **f)** R_u^n **g)** D_n^n **h)** S_2^n .
3. Let A be an $n \times n$ square matrix and $\vec{v} \in \mathbb{R}^n$ such that $A\vec{v} = \vec{0}$, prove that A is a degenerate matrix.
4. Find the eigenvalues and the eigenvectors of the maps D_n, S_2, R_u .
5. Find a real 2×2 matrix A such that $A^2 = -Id$.
Definition. Let B and D be two bases of the vector space V. Then P_{BD} is the matrix of changing from D-coordinates to B-coordinates, which means that for any $\vec{v} \in V$ holds $[\vec{v}]_B = P_{BD}[\vec{v}]_D$.
6. Let $B = \{1, x, x^2\}, D = \{1, x-1, (x-1)^2\}$ in $\mathbb{R}_2[x]$. Find P_{BD} .
7. Prove that **a)** P_{BD} is always nonsingular, **b)** $P_{BD} P_{DB} = I$ (here I is the *identity* or *unit* matrix).
Lemma 8. Let A_B and A_D be the matrices of the same linear map in the bases B and D respectively. Then $A_D = P_{DB} A_B P_{BD}$.
Definition. The matrices T and S are *similar* if there is nonsingular matrix P such that $T = PSP^{-1}$
Lemma 9. The matrices of the same linear map in different bases are similar.
Theorem 10. If T and S are similar then **a)** det(T)=det(S), **b)** tr(T)=tr(S), **c)** T and S have the same set of eigenvalues, **d)** the characteristic polynomials of T and S are the same.
Theorem 11. (*without proof*) Let A be a $n \times n$ matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$. Then A is similar to an upper triangular matrix with the entries $\lambda_1, \ldots, \lambda_n$ on the main diagonal.
12. Let A be a $n \times n$ matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$. Prove that eigenvalues of A^2 are $\lambda_1^2, \ldots, \lambda_n^2$.
Theorem 13. (*without proof*) Let S be a real symmetric matrix. Then
a) S is similar to a real diagonal matrix,
b) There is an orthogonal real basis consisting of S's eigenvectors.
14. Does there exist a real 3×3 matrix A such that tr(A) = 0 and $A^2 + A = Id$?
15 Given linear map $T : \mathbb{R}^2 \to \mathbb{R}^2$ and nonzero vectors $\vec{u}$ and $\vec{v}$ such that $\vec{u} \perp \vec{v}$ and $T(\vec{u}) = 2\vec{u}, T(\vec{v}) = 3\vec{v}$, prove that $|T(\vec{w})| \geq 2|\vec{w}|$.
IMC11.1.2. Does there exist a real 3×3 matrix A such that tr(A) = 0 and $A^2 + A^T = Id$?
IMC13.1.1. Let A and B be real symmetric matrices with all eigenvalues strictly greater than 1. Let u be a real eigenvalue of matrix AB. Prove that $|u| > 1$.
IMC14.2.2. Let A be a symmetric $n \times n$ matrix with real entries, let $a_{1,} a_{2,} \ldots, a_n$ denote values along its main diagonal, and let $l_{1,} l_{2,} \ldots, l_n$ denote its eigenvalues. Show that $\sum_{1 \leq i < j \leq n} a_i a_j \geq \sum_{1 \leq i < j \leq n} l_i l_j$ and determine all matrices for which the equality holds.
IMC7.1.3'. Prove that for any three 2×2 matrices A_1, A_2, A_3 , the polynomial $P(x_1, x_2, x_3) = \det(x_1 A_1 + x_2 A_2 + x_3 A_3)$ is not identical to $x_1^2 + x_2^2 + x_3^2$.