Polynomials and Equations

Facts and theorems

Theorem I (Bézout). Let a be a root of the polynomial $P(x)$. Then there exists a polynomial $Q(x)$ such that $P(x)=(x-a)Q(x)$.

Theorem II. a) Any polynomial of degree n has at most n roots.
b) If two polynomials of degree $\le n$ coincide in $n+1$ points they are identical.

Theorem III. Any polynomial of an odd degree with real coefficients has a real root.

Theorem IV (Viète). a) Numbers a and b are two roots of $x^2+px+q=0 \Leftrightarrow p=-(a+b),\ q=ab$.
b) Numbers a, b and c are three roots of $x^3+px^2+qx+r=0 \Leftrightarrow p=-(a+b+c),\ q=ab+ac+bc,\ r=-abc$.

Theorem V. Given are distinct numbers $x_1, x_2, \ldots, x_n$. Then for any numbers $y_1, y_2, \ldots, y_n$ there exist a unique polynomial $P(x)$ of degree less then n such that $P(x_1)=y_1, P(x_2)=y_2, \ldots, P(x_n)=y_n$.

Problems

1. Show without algebraic calculations that the polynomials $(x-1)(x-2)(x-3)(x-4)$ and $(x+1)(x+2)(x+3)(x+4)$ are not identical.

2. Can the product of two nonzero polynomials be equal to the zero polynomial?

3. In the polynomial identity $(x^2+4x+3)(2x-\ldots) = 2(x-1)(x+3)(x+\ldots)$ two numbers were replaced by dots. Restore the numbers.

4. Polynomial $(2x-5)^{100}=a_{100}x^{100}+a_{99}x^{99}+\ldots+a_1x+a_0$. Find the sum $a_0-a_1+a_2\ldots-a_{99}+a_{100}$.

5. Given are two polynomials $P(x)$ and $Q(x)$ with real coefficients. Their sets of roots are A and B, respectively. Construct a polynomial with the set of roots equal to **a**) $A\cup B$ **b**) $A\cap B$

6. How many roots has the equation $\frac{(x-2)(x-5)}{(9-2)(9-5)}+\frac{(x-9)(x-5)}{(2-9)(2-5)}+\frac{(x-9)(x-2)}{(5-9)(5-2)}=1$?

7. Given a polynomial $P(x)$. Construct a polynomial $Q(x)$ such that (a is a root of P <=> $3a$ is a root of Q).

8. Factorize the polynomials: **a)** x^3+x^2+2x-4 **b)** x^4+3x^2+4 **c)** x^4+9.

9. The product of four consecutive integers is 1680. Find these integers. How many solutions has the problem?

10. Given is a polynomial P. It is known that the equation $P(x)=x$ has no roots. Prove that the equation $P(P(x))=x$ has no roots.

Credit problems

PE1. It is known that the polynomials x^4+ax^2+1 and x^4+ax^3+1 have a common root. Find a.
PE2. Let $P=(x+1)^{101}(x+2)^{102}(x+3)^{103}=a_{306}x^{306}+a_{305}x^{305}+\ldots+a_1x+a_0$. Find the sum $a_0+a_2+\ldots+a_{306}$.
PE3. Let $P=x^4+x^3-3x^2+x+2$. Prove that for any positive integer $n>1$, the polynomial P^n has at least one negative coefficient.
PE4. Given are two polynomials P and Q, both are of positive degree. It is known that the identities $P(P(x)) \equiv Q(Q(x))$ and $P(P(P(x))) \equiv Q(Q(Q(x)))$ hold for any x. Does that imply the identity $P(x) \equiv Q(x)$?
PE5. For a reduced quadratic polynomial P with real coefficients let $M=\{x\in \mathbf{R}: |P(x)|<1\}$. Clearly M is either empty, or is an open interval, or two disjoint open intervals. Denote the sum of their lengths by $|M|$.
Prove that $|M|^2\le 8$.