Functional Equations

Definition. Let $\mathbb{R}_{pos}$ be the set of all positive real numbers.

Substituting numbers

1. The function $f:\mathbb{R}\to\mathbb{R}$ satisfies $f(x+y)=f(x)+f(y)+80xy \quad \forall x,y\in\mathbb{R}$.
a) Find $f(0.5)$, if $f(0.25) = 2.5$.
b) Find $f(6)$, if $f(1.5)=90$.

2. The function $f:\mathbb{R}\to\mathbb{R}$ satisfies $f(3x+1)=f(x)+3x+1{,}5 \quad \forall x\in\mathbb{R}$. It is known that $f(40)=67.5$. Find **a)** $f(13)$, **b)** $f(1)$.

3. The function $f:\mathbb{R}\to\mathbb{R}$ satisfies $f(xy)=f(x)f(y)-f(x)-y^2+1 \quad \forall x,y\in\mathbb{R}$.
Find $f(0)$.

4. Does there exist a function $f:\mathbb{R}_{pos}\to\mathbb{R}$, such that $f(xy)=4f(x)-3f(y)+2$ for every positive numbers x and y ?

5. The function $f:\mathbb{R}_{pos}\to\mathbb{R}$ satisfies $f(xy)=f(x)+f(y)$ for every positive numbers x and y. Find $f(20)$, if $f(0.05) = 10$.

6. The function $f:\mathbb{R}_{pos}\to\mathbb{R}_{pos}$ is such that $f(1)+f(2) = 10$ and
$f(x+y)=f(x)+f(y)+2\sqrt{f(x)f(y)} \quad \forall x,y\in\mathbb{R}_{pos}$. Find $f(2^{1000})$.

7. The function $f:\mathbb{R}\to\mathbb{R}$ is defined $\forall x\neq 1$ and $(x-1)f\left(\dfrac{x+1}{x-1}\right)=x+f(x)$. Find $f(-1)$.

Substituting expressions

8. Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that $f(x+y)+f(x-y)=x^2+y^2 \quad \forall x,y\in\mathbb{R}$.

9. Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that $f(2x+1)=4x^2+14x+7 \quad \forall x\in\mathbb{R}$

10. Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that $f(x)=x^2+2f(1-x) \quad \forall x\in\mathbb{R}$

11. The function $f:\mathbb{R}\to\mathbb{R}$ is defined $\forall x\neq 0$ and satisfies
$f(x)+f\left(\dfrac{x-1}{x}\right)=2x \quad \forall x\in\mathbb{R}, x\neq 0$. Find $f(x)$.

Auxiliary function

12. Find all functions $f:\mathbb{R}_{pos}\to\mathbb{R}$ such that $f(xy)+f(\frac{x}{y})=(\ln(x))^2+(\ln(y))^2 \quad \forall x,y\in\mathbb{R}_{pos}$

Credit problems

FE1. Find every $a\in\mathbb{R}$ such that there exists a non-constant function $f:\mathbb{R}\to\mathbb{R}$ satisfying $f(a(x+y))=f(x)+f(y)$.

FE2. Given the function $f:\mathbb{R}\to\mathbb{R}$ such that $(f(x))^2\leq f(y) \quad \forall x>y$, prove that $0\leq f(x)\leq 1 \quad \forall x\in\mathbb{R}$

FE3. Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that $f(x+y)+f(x-y)=2f(x)\cos(y) \quad \forall x,y\in\mathbb{R}$

FE4. Find all functions $f:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ such that $f(x,x)=0$ and $f(x,f(y,z))=f(x,y)+z \quad \forall x,y,z\in\mathbb{R}$

FE5. Find all functions $f:\mathbb{R}_{pos}\to\mathbb{R}$ such that $f(x^y)=f(x)^{f(y)} \quad \forall x,y\in\mathbb{R}_{pos}$