Integral and Differential Inequalities

(Integrate inequality) Use $f(x)>g(x)\Rightarrow\int f>\int g$

1. Given a real function $f(x)>x \quad \forall x$, prove that $\int\limits_0^1 f(x)\,dx>0.5$

2. Prove that **a)** $\left|\int\limits_a^{2a}\sin x^2\,dx\right|<|a| \quad \forall a\in\mathbb{R}$ **b)** $\int\limits_0^1 \sin x^2\,dx<\frac{1}{3}$ **c)** $\int\limits_0^1 \sin x^2\,dx>\frac{13}{42}$

3. $|f'(x)|<5 \quad \forall x.$ Prove that $|f(77)-f(7)|<350$.

4. Given f such that $f(0)=0$, f' be continuous and $f'(x)>\cos(x) \quad \forall x>0$, prove that $f(x)>\sin(x) \quad \forall x>0$.

5. $f(x)f'(x)\geq x \quad \forall x.$ Prove that $f^2(x)-f^2(y)\geq x^2-y^2 \quad \forall x>y$

6. Given f such that $f(2)=4$, f' be continuous and $f'(x)>\ln x+\frac{1}{\ln x} \quad \forall x>1$, prove that $f(x)>2x \quad \forall x>2$.

(Substitute) Use another function

7. Given f such that $f(0)=3$, f' be continuous and $f'(x)+2f(x)\geq 0 \quad \forall x$, prove that $f(x)<3e^{-2x} \quad \forall x<0$.

(Split segment) $\int\limits_a^b f+\int\limits_b^c f=\int\limits_a^c f$

8. $f(x)$ be continuous, $f(x)-x$ increasing. Prove that

$$\int\limits_{a+k}^{b+k} f(x)\,dx-\int\limits_a^b f(x)\,dx\geq k(b-a) \quad \forall b>a,\, k>0$$

9. Prove that **a)** $\sum\limits_{i=1}^{n}\frac{1}{i\sqrt[3]{i}}<4-\frac{1}{\sqrt[3]{n}}$ **b)** $\sum\limits_{i=2}^{\infty}\frac{1}{i\ln i}$ is divergent.

Credit problems

IDI1. Let $0<a<b$. Prove that $\int\limits_a^b (x^2+1)e^{-x^2}\,dx\geq e^{-a^2}-e^{-b^2}$.

IDI2. Call $f:[0,1]\to\mathbb{R}$ a *good* function if $|f(x)-f(y)|\geq|x-y|$ for all pairs $x,y\in\mathbb{R}$. Find the minimum of $\int\limits_0^1 f(x)\,dx$ over all good functions.

IDI3. Let $f:\mathbb{R}\to(0,+\infty)$ be a differentiable function, and suppose that there exists a constant L > 0 such that $|f'(x)-f'(y)|\leq L|x-y|$ for all pairs $x,y\in\mathbb{R}$. Prove that $(f'(x))^2<2Lf(x)$.

IDI4. Given f be two times differentiable such that $f(0)=1$, $f'(0)=0$ and $f''(x)-5f'(x)+6f(x)\geq 0 \quad \forall x\geq 0$, prove that $f(x)\geq 3e^{2x}-2e^{3x} \quad \forall x\geq 0$.

IDI5. Let $f:\mathbb{R}\to(0,+\infty)$ be a continuously differentiable function. Prove that

$$\left|\int\limits_0^1 f^3(x)\,dx-f^2(0)\int\limits_0^1 f(x)\,dx\right|\leq \max_{0\leq x\leq 1}|f'(x)|\left(\int\limits_0^1 f(x)\,dx\right)^2.$$